# Existance of ordinal ultrafilters and of $\mathcal{P}$-hierarchy.

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**Ordinal ultrafilters** (J. E. Baumgartner 1995) An ultrafilter $u$ is $J_\alpha$-**ultrafilter** if for each function $f : \omega \to \omega_1$ there is $U \in u$ such that $ot(f(U)) < \alpha$. **Proper** $J_\alpha$-**ultrafilters** are such $J_\alpha$-ultrafilters that are not $J_\beta$-ultrafilters for any $\beta < \alpha$. The class of proper $J_\alpha$ ultrafilters we denote by $J^*_\alpha$.

If $(u_n)_{n<\omega}$ is a sequence of filters on $\omega$ and $v$ is a filter on $\omega$ then the **contour on the sequence** $(u_n)$ **with respect to** $v$ is:

$$\int_v u_n = \bigcup_{V \in v} \bigcap_{n \in V} u_n$$

(Dolecki, Mynard 2002) **Monotone sequential contours** of rank 1 are exactly Frechet filters on infinite subsets of $\omega$; $u$ is a monotone sequential contours of rank $\alpha$ if $u = \int_{Fr} u_n$, where $(u_n)_{n<\omega}$ is a discrete sequence of monotone sequential contours such that: $r(u_n) \leq r(u_{n+1})$ and $\lim_{n<\omega}(r(u_n)+1) = \alpha$.

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Define classes $\mathcal{P}_\alpha$ of **P-hierarchy** for $\alpha < \omega_1$ as follows: $u \in \mathcal{P}_\alpha$ if there is no monotone sequential contour $c_\alpha$ of rank $\alpha$ such that $c_\alpha \subset u$, and for each $\beta < \alpha$ there exists a monotone sequential contour $c_\beta$ of rank $\beta$ such that $c_\beta \subset u$.

## Theorem (Baumgartner 1995)

*If P-points exist, then for each countable ordinal $\alpha$ the class of proper $J_{\omega^{\alpha+1}}$-ultrafilters is nonempty.*

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## Theorem ($MA_{\sigma-center}$)

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*For each countable ordinal $\alpha$ the class $P_{\alpha+\omega}$ is nonempty.*

**Theorem (CH)**

*For each countable ordinal $\alpha$ the class $P_{\alpha}$ is nonempty.*

**Theorem**

*Let $\alpha$ be limit ordinal and let $(v_n)$ be an increasing sequence of monotone sequential contours of rank less then $\alpha$ then $\bigcup_{n<\omega} v_n$ do not contain any monotone sequential contour of rank $\alpha$.*

**Theorem (ZFC)**

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The **cascade** is a tree $V$, ordered by "$\sqsubseteq$", without infinite branches and with a least element $\emptyset_V$. A cascade is **sequential** if for each non-maximal element of $V$ ($v \in V \setminus \max V$) the set $v^+$ of immediate successors of $v$ is countably infinite. If $v \in V \setminus \max V$, then the set $v^+$ may be endowed with an order of the type $\omega$, and then by $(v_n)_{n\in\omega}$ we denote the sequence of elements of $v^+$.
The **rank** of $v \in V$ ($r(v)$) is defined inductively: $r(v) = 0$ if $v \in \max V$, and otherwise $r(v) = \sup\{r(v_n) + 1 : n < \omega\}$. The rank $r(V)$ of the cascade $V$ is, by definition, the rank of $\emptyset_V$. If it is possible to order all sets $v^+$ (for $v \in V \setminus \max V$) so that for each $v \in V \setminus \max V$ the sequence $(r(v_n)_{n<\omega})$ is non-decreasing, then the cascade $V$ is **monotone**, and we fix such an order on $V$ without indication.

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We introduce **lexicographic order** $<_{lex}$ on $V$ in the following way: $v' <_{lex} v''$ if $v' \sqsupset v''$ or if there exist $v$, $\tilde{v'} \sqsubseteq v'$ and $\tilde{v''} \sqsubseteq v''$ such that $\tilde{v'} \in v^+$ and $\tilde{v''} \in v^+$ and $\tilde{v'} = v_n$, $\tilde{v''} = v_m$ and $n < m$. Also we label elements of a cascade $V$ by sequences of naturals of length $r(V)$ or less, by the function which preserves the lexicographic order, $v_l$ is a resulting name for an element of $V$, where $l$ is the mentioned sequence (i.e. $v_{l \frown n} = (v_l)_n \in v^+$); denote also $V_l = \{v \in V : v \sqsupseteq v_l\}$ and by $L_{\alpha,V}$ we understand $\{l \in \omega^{<\omega} : r_V(v_l) = \alpha\}$.

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Let $W$ be a cascade, and let $\{V_w : w \in \max W\}$ be a set of pairwise disjoint cascades such that $V_w \cap W = \emptyset$ for all $w \in \max W$. Then, the **confluence** of cascades $V_w$ with respect to the cascade $W$ (we write $W \looparrowleft V_w$) is defined as a cascade constructed by the identification of $w \in \max W$ with $\emptyset_{V_w}$ and according to the following rules: $\emptyset_W = \emptyset_{W \looparrowleft V_w}$; if $w \in W \setminus \max W$, then $w^{+W \looparrowleft V_w} = w^{+W}$; if $w \in V_{w_0}$ (for a certain $w_0 \in \max W$), then $w^{+W \looparrowleft V_w} = w^{+V_{w_0}}$; in each case we also assume that the order on the set of successors remains unchanged.

For the sequential cascade of rank 1, **contour** of $V$ is Frechet filter on $\max V$; $\int(V_0 \looparrowleft V_n) = \int_{\int V_0} \int V_n$.

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For a monotone sequential cascade $V$ by $f_V$ we denote an **lexicographic order respecting function** $\max V \to \omega_1$.

Let $V$ and $W$ be monotone sequential cascades such that $\max V \supset \max W$. We say that $W$ **increases the order of** $V$ (we write) $W \Rrightarrow V$ if $\mathrm{ot}\,(f_W(U)) \geq \mathrm{indec}\,(\mathrm{ot}\,(f_V(U)))$ for each $U \subset \max W$, where $\mathrm{indec}\,(\alpha)$ is the biggest indecomposable ordinal less then, or equal to $\alpha$;

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Let $u$ be an ultrafilter and let $V$, $W$ be monotone sequential cascades such that $\int V \subset u$ and $\int W \subset u$. Then we say that **rank** $\alpha$ **in cascade** $V$ **agree with rank** $\beta$ **in cascade** $W$ **with respect to the ultrafilter** $u$ if for any choice of $\tilde{V}_{p,s} \in \int V_p$ and $\tilde{W}_{p,s} \in \int W_s$ there is: $\bigcup_{(p,s)\in L_{\alpha,V}\times L_{\beta,W}}(\tilde{V}_{p,s} \cap \tilde{W}_{p,s}) \in u$; this relation is denoted by $\alpha_V E_u \beta_W$.

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*Let $u$ be an ultrafilter and let $V$, $W$ be monotone sequential cascades such that $\int V \subset u$ and $\int W \subset u$, then $1_V E_u 1_W$.*

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*Let $u$ be an ultrafilter and let $V$, $W$ be monotone sequential cascades of finite ranks such that $\int V \subset u$ and $\int W \subset u$. Then $n_V E_u m_W$ implies the existence of a monotone sequential cascade $T$ of rank $\max\{r(V), r(W)\} \leq r(T) \leq r(V) + r(W)$ and such that $\int T \subset u$ and $T \Rrightarrow V$ and $T \Rrightarrow W$.*

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