Existence of ordinal ultrafilters and of $\mathcal{P}$-hierarchy.

Andrzej Starosolski

Institute of Mathematics, Silesian University of Technology

Winter School in abstract Analysis section Set Theory, Hejnice
2011
Ordinal ultrafilters (J. E. Baumgartner 1995) An ultrafilter $u$ is $J_\alpha$-ultrafilter if for each function $f : \omega \to \omega_1$ there is $U \in u$ such that $ot(f(U)) < \alpha$. Proper $J_\alpha$-ultrafilters are such $J_\alpha$-ultrafilters that are not $J_\beta$-ultrafilters for any $\beta < \alpha$. The class of proper $J_\alpha$ ultrafilters we denote by $J_\alpha^\ast$. 
If \((u_n)_{n<\omega}\) is a sequence of filters on \(\omega\) and \(v\) is a filter on \(\omega\) then the **contour on the sequence** \((u_n)\) **with respect to** \(v\) is:

\[
\int_v u_n = \bigcup_{V \in v} \bigcap_{n \in V} u_n
\]

(Dolecki, Mynard 2002) **Monotone sequential contours** of rank 1 are exactly Frechet filters on infinite subsets of \(\omega\); \(u\) is a monotone sequential contours of rank \(\alpha\) if \(u = \int_{Fr} u_n\), where \((u_n)_{n<\omega}\) is a discrete sequence of monotone sequential contours such that: \(r(u_n) \leq r(u_{n+1})\) and \(\lim_{n<\omega} (r(u_n) + 1) = \alpha\).
If \((u_n)_{n<\omega}\) is a sequence of filters on \(\omega\) and \(v\) is a filter on \(\omega\) then the **contour on the sequence** \((u_n)\) **with respect to** \(v\) is:

\[
\int_v u_n = \bigcup_{V \in v} \bigcap_{n \in V} u_n
\]

(Dolecki, Mynard 2002) **Monotone sequential contours** of rank 1 are exactly Frechet filters on infinite subsets of \(\omega\); \(u\) is a monotone sequential contours of rank \(\alpha\) if \(u = \int_{Fr} u_n\), where \((u_n)_{n<\omega}\) is a discrete sequence of monotone sequential contours such that: 

\[r(u_n) \leq r(u_{n+1})\] and 
\[\lim_{n<\omega} (r(u_n) + 1) = \alpha.\]
Define classes $\mathcal{P}_\alpha$ of \textbf{P-hierarchy} for $\alpha < \omega_1$ as follows: $u \in \mathcal{P}_\alpha$ if there is no monotone sequential contour $c_\alpha$ of rank $\alpha$ such that $c_\alpha \subset u$, and for each $\beta < \alpha$ there exists a monotone sequential contour $c_\beta$ of rank $\beta$ such that $c_\beta \subset u$. 
### Theorem (Baumgartner 1995)

*If $P$-points exist, then for each countable ordinal $\alpha$ the class of proper $\mathcal{J}_{\omega^{\alpha+1}}$-ultrafilters is nonempty.*

### Theorem

*If $P$-points exist, then for each countable ordinal $\alpha$ the class $\mathcal{P}_{\alpha+1}$ is nonempty.*

### Theorem

*If $P$-points exist, then $\mathcal{P}_{\alpha+1} \cap \mathcal{J}^*_{\omega^{\alpha+1}} \neq \emptyset$ for each countable $\alpha$.***

### Theorem ($MA_{\sigma\text{-center}}$)

$\mathcal{P}_{\alpha+1} \neq \mathcal{J}^*_{\omega^{\alpha+1}}$ for $1 < \alpha < \omega_1$
Theorem (Baumgartner 1995)

If P-points exist, then for each countable ordinal $\alpha$ the class of proper $J_{\omega^{\alpha+1}}$-ultrafilters is nonempty.

Theorem

If P-points exist, then for each countable ordinal $\alpha$ the class $P_{\alpha+1}$ is nonempty.

Theorem

If P-points exist, then $P_{\alpha+1} \cap J^*_{\omega^{\alpha+1}} \neq \emptyset$ for each countable $\alpha$.

Theorem ($MA_{\sigma-center}$)

$P_{\alpha+1} \neq J^*_{\omega^{\alpha+1}}$ for $1 < \alpha < \omega_1$
Theorem (Baumgartner 1995)

*If P-points exist, then for each countable ordinal \( \alpha \) the class of proper \( J_{\omega^{\alpha+1}} \)-ultrafilters is nonempty.*

Theorem

*If P-points exist, then for each countable ordinal \( \alpha \) the class \( P_{\alpha+1} \) is nonempty.*

Theorem

*If P-points exist, then \( P_{\alpha+1} \cap J^*_{\omega^{\alpha+1}} \neq \emptyset \) for each countable \( \alpha \).*

Theorem (*MA*\( \sigma \)-center)

*\( P_{\alpha+1} \neq J^*_{\omega^{\alpha+1}} \) for \( 1 < \alpha < \omega_1 \)*
Theorem (Baumgartner 1995)

If P-points exist, then for each countable ordinal $\alpha$ the class of proper $J_{\omega^{\alpha+1}}$-ultrafilters is nonempty.

Theorem

If P-points exist, then for each countable ordinal $\alpha$ the class $\mathcal{P}_{\alpha+1}$ is nonempty.

Theorem

If P-points exist, then $\mathcal{P}_{\alpha+1} \cap J^{*}_{\omega^{\alpha+1}} \neq \emptyset$ for each countable $\alpha$.

Theorem ($MA_{\sigma-center}$)

$\mathcal{P}_{\alpha+1} \neq J^{*}_{\omega^{\alpha+1}}$ for $1 < \alpha < \omega_1$
### Theorem (Baumgartner 1995)

Let $1 < \alpha < \omega_1$ and assume $u$ is a proper $J_{\omega^{\alpha+2}}$-ultrafilter. Then there is a $P$-point ultrafilter $v$ such that $v \leq_{RK} u$.

### Theorem

Let $1 < \alpha < \omega_1$ is an ordinal and $u \in \mathcal{P}_{\alpha+1}$. Then there is a $P$-point ultrafilter $v$ such that $v \leq_{RK} u$.

### Theorem (Laflamme 1996) ($\text{MA}_{\sigma-\text{centr}}$)

There is proper $J_{\omega^{\omega+1}}$-ultrafilter all of whose RK-predecessors are proper $J_{\omega^{\omega+1}}$-ultrafilters.
**Theorem (Baumgartner 1995)**

Let $1 < \alpha < \omega_1$ and assume $u$ is a proper $J_{\omega^\alpha+2}$-ultrafilter. Then there is a $P$-point ultrafilter $v$ such that $v \leq_{RK} u$.

**Theorem**

Let $1 < \alpha < \omega_1$ is an ordinal and $u \in P_{\alpha+1}$. Then there is a $P$-point ultrafilter $v$ such that $v \leq_{RK} u$.

**Theorem (Laflamme 1996) \((MA_{\sigma-\text{centr}})\)**

There is proper $J_{\omega^{\omega+1}}$-ultrafilter all of whose $RK$-predecessors are proper $J_{\omega^{\omega+1}}$-ultrafilters.
Theorem (Baumgartner 1995)

Let \( 1 < \alpha < \omega_1 \) and assume \( u \) is a proper \( J_{\omega \alpha + 2} \)-ultrafilter. Then there is a \( P \)-point ultrafilter \( v \) such that \( v \leq_{RK} u \).

Theorem

Let \( 1 < \alpha < \omega_1 \) is an ordinal and \( u \in \mathcal{P}_{\alpha + 1} \). Then there is a \( P \)-point ultrafilter \( v \) such that \( v \leq_{RK} u \).

Theorem (Laflamme 1996) (MA\(_{\sigma \text{-centr}}\))

There is proper \( J_{\omega \omega + 1} \)-ultrafilter all of whose \( RK \)-predecessors are proper \( J_{\omega \omega + 1} \)-ultrafilters.
Theorem (MA\textsubscript{\textsigma−centr})

For each countable ordinal \( \alpha \) the class \( P_{\alpha+\omega} \) is nonempty.

Theorem (CH)

For each countable ordinal \( \alpha \) the class \( P_\alpha \) is nonempty.

Theorem

Let \( \alpha \) be limit ordinal and let \( (v_n) \) be an increasing sequence of monotone sequential contours of rank less then \( \alpha \) then \( \bigcup_{n<\omega} v_n \) do not contain any monotone sequential contour of rank \( \alpha \).

Theorem (ZFC)

The class of proper \( J_{\omega\omega} \)-ultrafilters is empty.
<table>
<thead>
<tr>
<th>Theorem (MA$_{\sigma-centr}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>For each countable ordinal $\alpha$ the class $P_{\alpha+\omega}$ is nonempty.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Theorem (CH)</th>
</tr>
</thead>
<tbody>
<tr>
<td>For each countable ordinal $\alpha$ the class $P_{\alpha}$ is nonempty.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let $\alpha$ be limit ordinal and let $(v_n)$ be an increasing sequence of monotone sequential contours of rank less then $\alpha$ then $\bigcup_{n&lt;\omega} v_n$ do not contain any monotone sequential contour of rank $\alpha$.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Theorem (ZFC)</th>
</tr>
</thead>
<tbody>
<tr>
<td>The class of proper $J_{\omega^\omega}$-ultrafilters is empty.</td>
</tr>
</tbody>
</table>
Theorem (MA$_{\sigma}$-centr)
For each countable ordinal $\alpha$ the class $P_{\alpha+\omega}$ is nonempty.

Theorem (CH)
For each countable ordinal $\alpha$ the class $P_{\alpha}$ is nonempty.

Theorem
Let $\alpha$ be limit ordinal and let $(v_n)$ be an increasing sequence of monotone sequential contours of rank less then $\alpha$ then $\bigcup_{n<\omega} v_n$ do not contain any monotone sequential contour of rank $\alpha$.

Theorem (ZFC)
The class of proper $J_{\omega^{\omega}}$-ultrafilters is empty.
Theorem (MA$_{\sigma-\text{centr}}$)

For each countable ordinal $\alpha$ the class $P_{\alpha+\omega}$ is nonempty.

Theorem (CH)

For each countable ordinal $\alpha$ the class $P_{\alpha}$ is nonempty.

Theorem

Let $\alpha$ be limit ordinal and let $(v_n)$ be an increasing sequence of monotone sequential contours of rank less then $\alpha$ then $\bigcup_{n<\omega} v_n$ do not contain any monotone sequential contour of rank $\alpha$.

Theorem (ZFC)

The class of proper $J_{\omega^\omega}$-ultrafilters is empty.
The **cascade** is a tree $V$, ordered by ”$\sqsubseteq$”, without infinite branches and with a least element $\emptyset_V$. A cascade is **sequential** if for each non-maximal element of $V$ ($v \in V \setminus \text{max } V$) the set $v^+$ of immediate successors of $v$ is countably infinite. If $v \in V \setminus \text{max } V$, then the set $v^+$ may be endowed with an order of the type $\omega$, and then by $(v_n)_{n \in \omega}$ we denote the sequence of elements of $v^+$. The **rank** of $v \in V$ ($r(v)$) is defined inductively: $r(v) = 0$ if $v \in \text{max } V$, and otherwise $r(v) = \sup \{r(v_n) + 1 : n < \omega\}$. The rank $r(V)$ of the cascade $V$ is, by definition, the rank of $\emptyset_V$. If it is possible to order all sets $v^+$ (for $v \in V \setminus \text{max } V$) so that for each $v \in V \setminus \text{max } V$ the sequence $(r(v_n))_{n<\omega}$ is non-decreasing, then the cascade $V$ is **monotone**, and we fix such an order on $V$ without indication.
The **cascade** is a tree $V$, ordered by "$\sqsubseteq$", without infinite branches and with a least element $\emptyset_V$. A cascade is **sequential** if for each non-maximal element of $V$ ($v \in V \setminus \text{max } V$) the set $v^+$ of immediate successors of $v$ is countably infinite. If $v \in V \setminus \text{max } V$, then the set $v^+$ may be endowed with an order of the type $\omega$, and then by $(v_n)_{n \in \omega}$ we denote the sequence of elements of $v^+$. The **rank** of $v \in V$ ($r(v)$) is defined inductively: $r(v) = 0$ if $v \in \text{max } V$, and otherwise $r(v) = \sup \{r(v_n) + 1 : n < \omega\}$. The rank $r(V)$ of the cascade $V$ is, by definition, the rank of $\emptyset_V$. If it is possible to order all sets $v^+$ (for $v \in V \setminus \text{max } V$) so that for each $v \in V \setminus \text{max } V$ the sequence $(r(v_n)_{n < \omega})$ is non-decreasing, then the cascade $V$ is **monotone**, and we fix such an order on $V$ without indication.
The **cascade** is a tree $V$, ordered by ”$\sqsubseteq$”, without infinite branches and with a least element $\emptyset_V$. A cascade is **sequential** if for each non-maximal element of $V$ ($v \in V \setminus \text{max } V$) the set $v^+$ of immediate successors of $v$ is countably infinite. If $v \in V \setminus \text{max } V$, then the set $v^+$ may be endowed with an order of the type $\omega$, and then by $(v_n)_{n \in \omega}$ we denote the sequence of elements of $v^+$.

The **rank** of $v \in V$ ($r(v)$) is defined inductively: $r(v) = 0$ if $v \in \text{max } V$, and otherwise $r(v) = \sup \{r(v_n) + 1 : n < \omega\}$. The rank $r(V)$ of the cascade $V$ is, by definition, the rank of $\emptyset_V$. If it is possible to order all sets $v^+$ (for $v \in V \setminus \text{max } V$) so that for each $v \in V \setminus \text{max } V$ the sequence $(r(v_n)_{n < \omega})$ is non-decreasing, then the cascade $V$ is **monotone**, and we fix such an order on $V$ without indication.
The **cascade** is a tree $V$, ordered by ”$\sqsubseteq$”, without infinite branches and with a least element $\emptyset_V$. A cascade is **sequential** if for each non-maximal element of $V$ ($v \in V \setminus \text{max } V$) the set $v^+$ of immediate successors of $v$ is countably infinite. If $v \in V \setminus \text{max } V$, then the set $v^+$ may be endowed with an order of the type $\omega$, and then by $(v_n)_{n \in \omega}$ we denote the sequence of elements of $v^+$. The **rank** of $v \in V$ ($r(v)$) is defined inductively: $r(v) = 0$ if $v \in \text{max } V$, and otherwise $r(v) = \sup \{r(v_n) + 1 : n < \omega\}$. The rank $r(V)$ of the cascade $V$ is, by definition, the rank of $\emptyset_V$. If it is possible to order all sets $v^+$ (for $v \in V \setminus \text{max } V$) so that for each $v \in V \setminus \text{max } V$ the sequence $(r(v_n)_{n<\omega})$ is non-decreasing, then the cascade $V$ is **monotone**, and we fix such an order on $V$ without indication.
The **cascade** is a tree $V$, ordered by ”$\sqsubseteq$”, without infinite branches and with a least element $\emptyset_V$. A cascade is **sequential** if for each non-maximal element of $V$ ($v \in V \setminus \text{max } V$) the set $v^+$ of immediate successors of $v$ is countably infinite. If $v \in V \setminus \text{max } V$, then the set $v^+$ may be endowed with an order of the type $\omega$, and then by $(v_n)_{n \in \omega}$ we denote the sequence of elements of $v^+$. The **rank** of $v \in V$ ($r(v)$) is defined inductively: $r(v) = 0$ if $v \in \text{max } V$, and otherwise $r(v) = \sup \{ r(v_n) + 1 : n < \omega \}$. The rank $r(V)$ of the cascade $V$ is, by definition, the rank of $\emptyset_V$. If it is possible to order all sets $v^+$ (for $v \in V \setminus \text{max } V$) so that for each $v \in V \setminus \text{max } V$ the sequence $(r(v_n)_{n < \omega})$ is non-decreasing, then the cascade $V$ is **monotone**, and we fix such an order on $V$ without indication.
We introduce **lexicographic order** $<_{\text{lex}}$ on $V$ in the following way: $v' <_{\text{lex}} v''$ if $v' \sqsubset v''$ or if there exist $v$, $\tilde{v}' \sqsubseteq v'$ and $\tilde{v}'' \sqsubseteq v''$ such that $\tilde{v}' \in v^+$ and $\tilde{v}'' \in v^+$ and $\tilde{v}' = v_n$, $\tilde{v}'' = v_m$ and $n < m$. Also we label elements of a cascade $V$ by sequences of naturals of length $r(V)$ or less, by the function which preserves the lexicographic order, $v_l$ is a resulting name for an element of $V$, where $l$ is the mentioned sequence (i.e. $v_{l \sim n} = (v_l)_n \in v^+$); denote also $V_l = \{ v \in V : v \sqsupseteq v_l \}$ and by $L_{\alpha, V}$ we understand $\{ l \in \omega^{<\omega} : r_V(v_l) = \alpha \}$. 
We introduce **lexicographic order** 
\(<_{lex}\) on \(V\) in the following way: 
\(v' <_{lex} v''\) if \(v' \sqsupseteq v''\) or if there exist \(v, \tilde{v}' \sqsubseteq v'\) and \(\tilde{v}'' \sqsubseteq v''\) such that \(\tilde{v}' \in v^+\) and \(\tilde{v}'' \in v^+\) and \(\tilde{v}' = v_n, \tilde{v}'' = v_m\) and \(n < m\). Also we label elements of a cascade \(V\) by sequences of naturals of length \(r(V)\) or less, by the function which preserves the lexicographic order, \(v_l\) is a resulting name for an element of \(V\), where \(l\) is the mentioned sequence (i.e. \(v_l \sim_n = (v_l)_n \in v^+\)); denote also \(V_l = \{v \in V : v \sqsupseteq v_l\}\) and by \(L_{\alpha, V}\) we understand \(\{l \in \omega^{<\omega} : r_V(v_l) = \alpha\}\).
We introduce **lexicographic order** $<_{\text{lex}}$ on $V$ in the following way: $v' <_{\text{lex}} v''$ if $v' \sqsupseteq v''$ or if there exist $v$, $\tilde{v}' \sqsubseteq v'$ and $\tilde{v}'' \sqsubseteq v''$ such that $\tilde{v}' \in v^+$ and $\tilde{v}'' \in v^+$ and $\tilde{v}' = v_n$, $\tilde{v}'' = v_m$ and $n < m$. Also we label elements of a cascade $V$ by sequences of naturals of length $r(V)$ or less, by the function which preserves the lexicographic order, $v_l$ is a resulting name for an element of $V$, where $l$ is the mentioned sequence (i.e. $v_l \sim_n = (v_l)_n \in v^+$); denote also $V_l = \{ v \in V : v \sqsupseteq v_l \}$ and by $L_{\alpha, V}$ we understand $\{ l \in \omega^{<\omega} : r_V(v_l) = \alpha \}$. 
We introduce **lexicographic order** \( <_{\text{lex}} \) on \( V \) in the following way: \( v' <_{\text{lex}} v'' \) if \( v' \sqsupseteq v'' \) or if there exist \( v, \tilde{v}' \sqsubseteq v' \) and \( \tilde{v}'' \sqsubseteq v'' \) such that \( \tilde{v}' \in v^+ \) and \( \tilde{v}'' \in v^+ \) and \( \tilde{v}' = v_n, \tilde{v}'' = v_m \) and \( n < m \). Also we label elements of a cascade \( V \) by sequences of naturals of length \( r(V) \) or less, by the function which preserves the lexicographic order, \( v_l \) is a resulting name for an element of \( V \), where \( l \) is the mentioned sequence (i.e. \( v_l \sim_n = (v_l)_n \in v^+ \)); denote also \( V_l = \{ v \in V : v \sqsupseteq v_l \} \) and by \( L_{\alpha, V} \) we understand \( \{ l \in \omega^{<\omega} : r_V(v_l) = \alpha \} \).
Let $\mathcal{W}$ be a cascade, and let $\{V_w : w \in \text{max } \mathcal{W}\}$ be a set of pairwise disjoint cascades such that $V_w \cap \mathcal{W} = \emptyset$ for all $w \in \text{max } \mathcal{W}$. Then, the confluence of cascades $V_w$ with respect to the cascade $\mathcal{W}$ (we write $\mathcal{W} \looparrowright V_w$) is defined as a cascade constructed by the identification of $w \in \text{max } \mathcal{W}$ with $\emptyset V_w$ and according to the following rules:

- $\emptyset \mathcal{W} = \emptyset \mathcal{W} \looparrowright V_w$;
- if $w \in \mathcal{W} \setminus \text{max } \mathcal{W}$, then $w^+ \mathcal{W} \looparrowright V_w = w^+ \mathcal{W}$;
- if $w \in V_{w_0}$ (for a certain $w_0 \in \text{max } \mathcal{W}$), then $w^+ \mathcal{W} \looparrowright V_w = w^+ V_{w_0}$;

in each case we also assume that the order on the set of successors remains unchanged.

For the sequential cascade of rank 1, the contour of $V$ is Frechet filter on $\text{max } V$; $\int (V_0 \looparrowright V_n) = \int V_0 \int V_n$. 
Let $W$ be a cascade, and let $\{V_w : w \in \text{max } W\}$ be a set of pairwise disjoint cascades such that $V_w \cap W = \emptyset$ for all $w \in \text{max } W$. Then, the confluence of cascades $V_w$ with respect to the cascade $W$ (we write $W \hookrightarrow V_w$) is defined as a cascade constructed by the identification of $w \in \text{max } W$ with $\emptyset V_w$ and according to the following rules: $\emptyset W = \emptyset W \hookrightarrow V_w$; if $w \in W \setminus \text{max } W$, then $w + W \hookrightarrow V_w = w + W$; if $w \in V_{w_0}$ (for a certain $w_0 \in \text{max } W$), then $w + W \hookrightarrow V_w = w + V_{w_0}$; in each case we also assume that the order on the set of successors remains unchanged.

For the sequential cascade of rank 1, the contour of $V$ is Frechet filter on $\text{max } V$; $\int (V_0 \hookrightarrow V_n) = \int \int V_0 \int V_n$. 

Andrzej Starosolski

Existence of ordinal ultrafilters and of $P$-hierarchy.
For a monotone sequential cascade $V$ by $f_V$ we denote an \textbf{lexicographic order respecting function} $\max V \to \omega_1$.

Let $V$ and $W$ be monotone sequential cascades such that $\max V \supset \max W$. We say that $W$ \textit{increases the order of} $V$ (we write) $W \Rightarrow V$ if $\ot (f_W(U)) \geq \text{indec} (\ot (f_V(U)))$ for each $U \subset \max W$, where $\text{indec} (\alpha)$ is the biggest indecomposable ordinal less than, or equal to $\alpha$;
For a monotone sequential cascade \( V \) by \( f_V \) we denote an **lexicographic order respecting function** \( \max V \to \omega_1 \).

Let \( V \) and \( W \) be monotone sequential cascades such that \( \max V \supset \max W \). We say that \( W \) **increases the order of** \( V \) (we write) \( W \Rightarrow V \) if \( \operatorname{ot} ( f_W (U)) \geq \operatorname{indec} \left( \operatorname{ot} \left( f_V (U) \right) \right) \) for each \( U \subset \max W \), where \( \operatorname{indec} (\alpha) \) is the biggest indecomposable ordinal less then, or equal to \( \alpha \);
Let $u$ be an ultrafilter and let $V$, $W$ be monotone sequential cascades such that $\int V \subset u$ and $\int W \subset u$. Then we say that \textbf{rank $\alpha$ in cascade $V$ agree with rank $\beta$ in cascade $W$ with respect to the ultrafilter $u$} if for any choice of $\tilde{V}_{p,s} \in \int V_p$ and $\tilde{W}_{p,s} \in \int W_s$ there is: $\bigcup_{(p,s) \in L_{\alpha,V} \times L_{\beta,W}} (\tilde{V}_{p,s} \cap \tilde{W}_{p,s}) \in u$; this relation is denoted by $\alpha_V E_u \beta_W$.
Theorem

Let $u$ be an ultrafilter and let $V, W$ be monotone sequential cascades such that $\int V \subset u$ and $\int W \subset u$, then $1_V E_u 1_W$.

Theorem

Let $u$ be an ultrafilter and let $V, W$ be monotone sequential cascades of finite ranks such that $\int V \subset u$ and $\int W \subset u$. Then $n_V E_u m_W$ implies the existence of a monotone sequential cascade $T$ of rank $\max \{r(V), r(W)\} \leq r(T) \leq r(V) + r(W)$ and such that $\int T \subset u$ and $T \Rightarrow V$ and $T \Rightarrow W$. 
Theorem

Let $u$ be an ultrafilter and let $V, W$ be monotone sequential cascades such that $\int V \subset u$ and $\int W \subset u$, then $1_V E_u 1_W$.

Theorem

Let $u$ be an ultrafilter and let $V, W$ be monotone sequential cascades of finite ranks such that $\int V \subset u$ and $\int W \subset u$. Then $n_V E_u m_W$ implies the existence of a monotone sequential cascade $T$ of rank $\max \{ r(V), r(W) \} \leq r(T) \leq r(V) + r(W)$ and such that $\int T \subset u$ and $T \Rightarrow V$ and $T \Rightarrow W$. 
A. Starosolski, P-hierarchy on $\beta\omega$, J. Symb. Log. 73, 4 (2008), 1202-1214.
A. Starosolski, Ordinal ultrafilters versus P-hierarchy, to appear.
A. Starosolski, Cascades, order and ultrafilters, to appear.