Forcing, ideals and degrees of reals

Marcin Sabok (KGRC, Wien)

Winterschool, 4 February 2010
Joint work

This is joint work with Jindra Zapletal
Idealized forcings

Given a $\sigma$-ideal $I$ on a Polish space $X$ we consider the forcing notion $P_I$ of $I$-positive Borel sets, ordered by inclusion.
Idealized forcings

Given a $\sigma$-ideal $I$ on a Polish space $X$ we consider the forcing notion $P_I$ of $I$-positive Borel sets, ordered by inclusion.

General question

A general question to ask is: what are the connections between descriptive set-theoretic properties of $I$ and forcing properties of $P_I$?
For example: properness

The forcing $P_I$ is proper if and only if for every countable $M \prec H_\kappa$ and $B \in P_I \cap M$ the set

$$\{ x \in B : x \text{ is } P_I\text{-generic over } M \}$$

is $I$-positive.
For example: properness

The forcing $P_I$ is proper if and only if for every countable $M \prec H_\kappa$ and $B \in P_I \cap M$ the set

$$\{x \in B : x \text{ is } P_I\text{-generic over } M\}$$

is $I$-positive.

Theorem (Zapletal, 2002)

If $I$ is generated by closed sets, then $P_I$ is proper.
Generating by closed sets

We use the following notation. Given a \( \sigma \)-ideal \( I \) on a Polish space \( X \) we write \( I^* \) for the \( \sigma \)-ideal generated by closed sets which belong to \( I \).
Generating by closed sets

We use the following notation. Given a $\sigma$-ideal $I$ on a Polish space $X$ we write $I^*$ for the $\sigma$-ideal generated by closed sets which belong to $I$.

General question

Another question is the following: are there any connections between forcing properties of $P_I$ and $P_{I^*}$. 

Marcin Sabok (KGRC, Wien)
Forcing, ideals and degrees of reals
Relatives of the Miller forcing

Given an ideal $J$ on $ω$ (or just a hereditary family of subsets of $ω$) we consider the forcing notion $Q(J)$ of all subtrees $T \subseteq ω^{<ω}$ with the following property:

- for each $τ ∈ T$ there is $σ ∈ T$ such that $τ ⊆ σ$ and the set of immediate successors of $σ$ in $T$ is $J$-positive.
Relatives of the Miller forcing

Given an ideal $J$ on $\omega$ (or just a hereditary family of subsets of $\omega$) we consider the forcing notion $Q(J)$ of all subtrees $T \subseteq \omega^{<\omega}$ with the following property:

- for each $\tau \in T$ there is $\sigma \in T$ such that $\tau \subseteq \sigma$ and the set of immediate successors of $\sigma$ in $T$ is $J$-positive.

Example

If $J$ is the family of finite subsets of $\omega$, then $Q(J)$ is just the Miller forcing.
Given an ideal $J$ on $\omega$ (or just a hereditary family of subsets of $\omega$) we consider the forcing notion $Q(J)$ of all subtrees $T \subseteq \omega^{<\omega}$ with the following property:

- for each $\tau \in T$ there is $\sigma \in T$ such that $\tau \subseteq \sigma$ and the set of immediate successors of $\sigma$ in $T$ is $J$-positive.

**Example**

If $J$ is the family of finite subsets of $\omega$, then $Q(J)$ is just the Miller forcing.

**Proposition (S.–Zapletal)**

If $J$ is a hereditary family of subsets of $\omega$, then $Q(J)$ is forcing equivalent to $P_{IJ}$ for some $\sigma$-ideal $I_J$ generated by closed sets.
Katětov order

Given two hereditary families $J$ and $J'$ of subsets of $\text{dom}(J)$ and $\text{dom}(J')$, respectively we say that $J$ is Katětov below $J'$, in symbols $J \leq_K J'$ if there is a map

$$f : \text{dom}(J') \rightarrow \text{dom}(J)$$

such that $f^{-1}(a) \in J'$ for each $a \in J$. 

Notice that if $J$ and $J'$ are ultrafilters, then this is just the Rudin–Keisler ordering.
Katětov order

Given two hereditary families $J$ and $J'$ of subsets of $\text{dom}(J)$ and $\text{dom}(J')$, respectively we say that $J$ is Katětov below $J'$, in symbols $J \leq_K J'$ if there is a map

$$f : \text{dom}(J') \to \text{dom}(J)$$

such that $f^{-1}(a) \in J'$ for each $a \in J$.

Notice

Notice that if $J$ and $J'$ are ultrafilters, then this is just the Rudin–Keisler ordering.
Katětov order

Given two hereditary families $J$ and $J'$ of subsets of $\text{dom}(J)$ and $\text{dom}(J')$, respectively we say that $J$ is Katětov below $J'$, in symbols $J \leq_K J'$ if there is a map

$$f : \text{dom}(J') \to \text{dom}(J)$$

such that $f^{-1}(a) \in J'$ for each $a \in J$.

Notice

Notice that if $J$ and $J'$ are ultrafilters, then this is just the Rudin–Keisler ordering.

Notation

If $J$ is a hereditary family of subsets of $\omega$ and $a \subseteq \omega$ is $J$-positive, then we write $J \upharpoonright a$ for the family of subsets of $a$ which are in $J$. 
Nowhere dense

We write $\text{NWD}$ for the ideal on $2^{<\omega}$ of those $A$ such that for any $\tau \in 2^{<\omega}$ there is $\sigma$ extending $\tau$ such that no further extension of $\sigma$ falls into $A$. 

Theorem (S.–Zapletal, 2009) The forcing $Q(J)$ does not add Cohen reals if and only if $J \upharpoonright a \not\geq K_{\text{NWD}}$ for any $J$-positive set $a$. 

Marcin Sabok (KGRC, Wien) 
Forcing, ideals and degrees of reals
Nowhere dense

We write NWD for the ideal on $2^{<\omega}$ of those $A$ such that for any $\tau \in 2^{<\omega}$ there is $\sigma$ extending $\tau$ such that no further extension of $\sigma$ falls into $A$.

Theorem (S.–Zapletal, 2009)

The forcing $Q(J)$ does not add Cohen reals if and only if

$$J \upharpoonright a \not\in_K NWD$$

for any $J$-positive set $a$. 
Fubini property

If \( a \subseteq \text{dom}(J) \) and \( D \subseteq a \times 2^\omega \), then we write

\[
\int_a D \ dJ = \{ y \in 2^\omega : \{ j \in a : \langle j, y \rangle \notin D \} \in J \}.
\]

\( J \) has the \textit{Fubini property} if for every real \( \varepsilon > 0 \), every \( J \)-positive set \( a \) and every Borel set \( D \subseteq a \times 2^\omega \) with vertical sections of Lebesgue measure less than \( \varepsilon \), the set \( \int_a D \ dJ \) has outer measure at most \( \varepsilon \).
Fubini property

If \( a \subseteq \text{dom}(J) \) and \( D \subseteq a \times 2^\omega \), then we write

\[
\int_a D \ dJ = \{ y \in 2^\omega : \{ j \in a : \langle j, y \rangle \notin D \} \in J \}.
\]

\( J \) has the **Fubini property** if for every real \( \varepsilon > 0 \), every \( J \)-positive set \( a \) and every Borel set \( D \subseteq a \times 2^\omega \) with vertical sections of Lebesgue measure less than \( \varepsilon \), the set \( \int_a D \ dJ \) has outer measure at most \( \varepsilon \).

Definition

Let \( 0 < \varepsilon < 1 \) be a real number. The ideal \( S_\varepsilon \) has as its domain all clopen subsets of \( 2^\omega \) of Lebesgue measure less than \( \varepsilon \), and it is generated by those sets \( a \) with \( \bigcup a \neq 2^\omega \).
Remark

Obviously, the ideals $S_ε$ as well as all families above them in the Katětov ordering fail to have the Fubini property.
Remark

Obviously, the ideals $S_\varepsilon$ as well as all families above them in the Katětov ordering fail to have the Fubini property.

Theorem (Solecki, 2000)

Suppose $F$ is an ideal on a countable set. Then either $F$ has the Fubini property, or else for every (or equivalently, some) $\varepsilon > 0$ there is a $F$-positive set $a$ such that

$$F \upharpoonright a \geq_K S_\varepsilon.$$
Recall

Recall that a forcing notion preserves outer Lebesgue measure if the set of ground model reals does not have measure zero in the generic extension.
Recall

Recall that a forcing notion preserves outer Lebesgue measure if the set of ground model reals does not have measure zero in the generic extension.

Theorem (S.–Zapletal, 2009)

Suppose that $J$ is a universally measurable ideal. $Q(J)$ preserves outer Lebesgue measure if and only if $J$ has the Fubini property.
Proof

Suppose on one hand that $J$ fails to have the Fubini property. Find a sequence of $J$-positive sets $\langle b_n : n \in \omega \rangle$ such that

$$J \restriction b_n \geq_K S_{2^{-n}},$$

as witnessed by functions $f_n$. Consider the tree $T$ of all sequences $t \in \text{dom}(J)^{<\omega}$ such that $t(n) \in b_n$ for each $n \in \text{dom}(t)$. Let $\dot{B}$ be a name for the set

$$\{ z \in 2^\omega : \exists \infty n \ z \in f_n(\dot{g}(n)) \}.$$

$T$ forces that the set $\dot{B}$ has measure zero, and the definition of the ideals $S_\varepsilon$ shows that every ground model point in $2^\omega$ is forced to belong to $\dot{B}$. Thus $Q(J)$ fails to preserve Lebesgue outer measure at least below the condition $T$. 
On the other hand, suppose that the ideal $J$ does have the Fubini property. Suppose that $Z \subseteq 2^\omega$ is a set of outer Lebesgue measure $\delta$, $\dot{O}$ is a $Q(J)$-name for an open set of measure less or equal to $\varepsilon < \delta$, and $T \in Q(J)$ is a condition. We must find a point $z \in Z$ and a condition $S \leq T$ forcing $\bar{z} \notin \dot{O}$.

By a standard fusion argument, thinning out the tree $T$ if necessary, we may assume that there is a function $h : \text{split}(T) \to O$ such that

$$T \forces \dot{O} = \bigcup \{h(\dot{g} \upharpoonright n + 1) : \dot{g} \upharpoonright n \in \text{split}(T)\}.$$

Moreover, we can make sure that if $t_n \in T$ is the $n$-th splitting node, then $T \upharpoonright t_n$ decides a subset of $\dot{O}$ with measure greater than $\varepsilon/2^n$. Hence, if we write $f(t_n) = \varepsilon/2^n$, then for every splitnode $t \in T$ and every $n \in \text{succ}_T(t)$ we have $\mu(h(t \cap n)) < f(t)$.
Proof

Now, for every splitnode $t \in T$ let

$$D_t = \{ \langle O, x \rangle : x \in 2^\omega \land O \in \text{succ}_T(t) \land x \in h(t \upharpoonright O) \}.$$ 

It follows from universal measurability of $J$ that the set

$$\int_{\text{succ}_T(t)} D_t \, dJ$$ 

is measurable. It has mass not greater than $f(t)$, by the Fubini assumption. Since $\sum_{t\in\text{split}(T)} f(t) < \delta$, we can find

$$z \in Z \setminus \bigcup_{t\in\text{split}(T)} \int_{\text{succ}_T(t)} D_t \, dJ.$$ 

Let $S \subseteq T$ be the downward closure of those nodes $t \upharpoonright n$ such that $t \in T$ is a splitnode and $n \in \text{succ}_T(t)$ is such that $z \notin h(t \upharpoonright n)$. $S$ belongs to $Q(J)$ by the choice of the point $z$ and $S \models \check{z} \notin \check{O}$, as required.
Definition

SPL is the family of nonsplitting subsets of $\omega^\omega$, i.e. those $a \subseteq \omega^\omega$ for which there is an infinite set $c \subseteq \omega$ such that $t \upharpoonright c$ is constant for every $t \in a$. 
**Definition**

SPL is the family of nonsplitting subsets of $\omega^<\omega$, i.e. those $a \subseteq \omega^<\omega$ for which there is an infinite set $c \subseteq \omega$ such that $t \upharpoonright c$ is constant for every $t \in a$.

**Theorem (S.–Zapletal, 2009)**

Suppose that $J$ is coanalytic hereditary, or an ideal with the Baire property. Then $Q(J)$ does not add splitting reals if and only if

$$J \upharpoonright a \not\subseteq_K SPL$$

for every $J$-positive $a$. 
**Definition**

SPL is the family of nonsplitting subsets of $\omega^\omega$, i.e. those $a \subseteq \omega^\omega$ for which there is an infinite set $c \subseteq \omega$ such that $t \upharpoonright c$ is constant for every $t \in a$.

**Theorem (S.–Zapletal, 2009)**

Suppose that $J$ is coanalytic hereditary, or an ideal with the Baire property. Then $Q(J)$ does not add splitting reals if and only if

$$J \upharpoonright a \notin_K SPL$$

for every $J$-positive $a$.

**Question**

Is SPL a Borel set?
Definition

We say that two reals \( x, y \) in a forcing extension have the same **continuous degree** if there is a **ground model partial homeomorphism** \( f \) of the real line (both domain and range should be \( G_\delta \)) such that

\[
f(x) = y.
\]

We say that a forcing **adds one continuous degree** if all reals in the extension have the same continuous degree.
Definition

We say that an ideal $J$ has the \textit{discrete set property} if it is not Katětov above the ideal generated by discrete subsets of the rationals.
Definition

We say that an ideal \( J \) has the\textit{ discrete set property} if it is not Katětov above the ideal generated by discrete subsets of the rationals.

Proposition (S.–Zapletal)

If \( J \) has the discrete set property, then \( Q(J) \) adds one continuous degree.
Definition

Fix a Polish space $X$ and its countable basis $\mathcal{O}$ of open sets. For a set $a \subseteq \mathcal{O}$, define

$$\text{cl}(a) = \{x \in X : \forall \varepsilon > 0 \exists O \in a \quad O \subseteq B_\varepsilon(x)\},$$

where $B_\varepsilon(x)$ stands for the ball centered at $x$ with radius $\varepsilon$. For a $\sigma$-ideal $I$ on $X$ we write

$$J_I = \{a \subseteq \mathcal{O} : \text{cl}(a) \in I\}.$$
Definition

Fix a Polish space $X$ and its countable basis $\mathcal{O}$ of open sets. For a set $a \subseteq \mathcal{O}$, define

$$\text{cl}(a) = \{ x \in X : \forall \varepsilon > 0 \ \exists O \in a \ \ O \subseteq B_\varepsilon(x) \},$$

where $B_\varepsilon(x)$ stands for the ball centered at $x$ with radius $\varepsilon$. For a $\sigma$-ideal $I$ on $X$ we write

$$J_I = \{ a \subseteq \mathcal{O} : \text{cl}(a) \in I \}.$$ 

Remark

Note that if $X$ is compact and $J_I$ is analytic, then it follows from the Kechris–Louveau–Woodin theorem that $J_I$ is $F_{\sigma\delta}$. 
Definition

An ideal $J$ on a countable set is \textit{weakly selective} if for every $J$-positive set $a$, any function on $a$ is either constant or 1-1 on a positive subset of $a$. 
Definition

An ideal $J$ on a countable set is weakly selective if for every $J$-positive set $a$, any function on $a$ is either constant or 1-1 on a positive subset of $a$.

Proposition (S.–Zapletal, 2009)

$J_I$ is weakly selective for any $\sigma$-ideal $I$. 
Theorem (Category Dichotomy, Hrušák, 2008)

If $I$ is a Borel ideal, then

- either $I \leq_{K} NWD$,
- or $ED \leq_{K} I \upharpoonright a$ for some $I$-positive $a$,

where $ED$ is the ideal on $\omega \times \omega$ generated by graphs of functions and vertical sections.
Theorem (Category Dichotomy, Hrušák, 2008)

If $I$ is a Borel ideal, then
- either $I \leq_K \text{NWD}$,
- or $ED \leq_K I \upharpoonright a$ for some $I$-positive $a$,

where $ED$ is the ideal on $\omega \times \omega$ generated by graphs of functions and vertical sections.

Corollary

If a Borel ideal $J$ is weakly selective, then $I \leq_K \text{NWD}$. 
Conjecture

If $J$ is a dense (tall) $F_{\sigma\delta}$ weakly selective ideal on $\omega$, then there exists a Polish space with a countable base $O$ and a $\sigma$-ideal $I$ on $X$ such that under some identification of $\omega$ and $O$ the ideal $J$ becomes $J_I$. 
Theorem (S.–Zapletal, 2009)
If \( I \) is a \( \sigma \)-ideal on a Polish space such that \( P_I \) is not equivalent to the Cohen forcing under any condition, then

\[
P_I^* \equiv Q(J_I).
\]
Theorem (S.–Zapletal, 2009)
If $I$ is a $\sigma$-ideal on a Polish space such that $P_I$ is not equivalent to the Cohen forcing under any condition, then

$$P_I^* \equiv Q(J_I).$$

Proposition (S.–Zapletal, 2009)
If $I$ is such that $P_I^*$ is equivalent to the Miller, Sacks or Cohen forcing, then $I = I^*$. 
Theorem (S.–Zapletal, 2009)

If $I$ is a $\sigma$-ideal such that $P_I$ is proper and does not add Cohen reals, then $J_I \upharpoonright a \not\geq K \text{ NWD}$ for any $J_I$-positive $a$. 

Corollary

If $I$ is such that $P_I$ is proper and does not add Cohen reals, then $P_I$ inherits these properties.
Theorem (S.–Zapletal, 2009)
If $I$ is a $\sigma$-ideal such that $P_I$ is proper and does not add Cohen reals, then $J_I \upharpoonright a \not\geq_K NWD$ for any $J_I$-positive $a$.

Corollary
If $I$ is such that $P_I$ is proper and does not add Cohen reals, then $P_I^*$ inherits these properties.
Theorem (S.–Zapletal, 2009)

If $I$ is a $\sigma$-ideal such that $P_I$ is proper and preserves outer Lebesgue measure, then $J_I$ has the Fubini property.
Theorem (S.–Zapletal, 2009)

If $I$ is a $\sigma$-ideal such that $P_I$ is proper and preserves outer Lebesgue measure, then $J_I$ has the Fubini property.

Corollary

If $I$ is $\Pi^1_1$ on $\Sigma^1_1$ such that $P_I$ is proper and preserves outer Lebesgue measure, then $P_I^*$ inherits these properties.
Theorem (S.–Zapletal, 2009)

If $I$ is such that $P_I$ is proper and $\omega^\omega$-bounding, then $J_I$ has the discrete set property.
Theorem (S.–Zapletal, 2009)

If $I$ is such that $P_I$ is proper and $\omega^\omega$-bounding, then $J_I$ has the discrete set property.

Corollary

If $I$ is such that $P_I$ is proper and $\omega^\omega$-bounding, then $P_I^*$ adds one continuous degree.
Definition

We say that two reals $x, y$ in a forcing extension have the same Borel degree if there is a ground model Borel automorphism $f$ of the real line such that

$$f(x) = y.$$ 

We say that a forcing adds one Borel degree if all reals in the extension have the same Borel degree.
Definition

We say that two reals $x, y$ in a forcing extension have the same Borel degree if there is a ground model Borel automorphism $f$ of the real line such that

$$f(x) = y.$$ 

We say that a forcing adds one Borel degree if all reals in the extension have the same Borel degree.

Remarks

Recall that the Sacks and Miller forcing add one Borel (or even continuous) degree. On the other hand, the Cohen forcing adds “many” Borel degrees.
Theorem (S.–Zapletal, 2009)

If $I$ is a $\sigma$-ideal generated by closed sets, then every real in the extension is either a ground model real, a Cohen real, or else has the same Borel degree as the generic real.
Theorem (S.–Zapletal, 2009)

If $I$ is a $\sigma$-ideal generated by closed sets, then every real in the extension is either a ground model real, a Cohen real, or else has the same Borel degree as the generic real.

Remark on the proof

The proof uses a technique motivated by the Gandy–Harrington topology. Instead of recursively coded analytic sets, we use the Borel sets coded inside a countable model $M$ as a topology base.
Theorem (Continuous reading of names, Zapletal, 2002)

If $I$ is a $\sigma$-ideal generated by closed sets, then every Borel map defined on a Borel $I$-positive set can be restricted to a Borel $I$-positive set, on which it is continuous.
Theorem (Continuous reading of names, Zapletal, 2002)

If $I$ is a $\sigma$-ideal generated by closed sets, then every Borel map defined on a Borel $I$-positive set can be restricted to a Borel $I$-positive set, on which it is continuous.

Theorem (S.–Zapletal, 2009)

If $I$ is a $\sigma$-ideal generated by closed sets, then the following are equivalent:

- $P_I$ does not add Cohen reals,
- every Borel map defined on a Borel $I$-positive set can be restricted to a Borel $I$-positive set, on which it is either 1-1 or constant.
Question
Suppose $I$ is generated by closed sets and $P_I$ does not add Cohen reals. Does $P_I$ necessarily add one *continuous* degree?
Theorem (Mycielski, ?)

If $A \subseteq 2^\omega \times 2^\omega$ is analytic such that all sections $A_x$ are countable, then there is a perfect set which is free for $A$. 
Theorem (Mycielski, ?)  
If $A \subseteq 2^\omega \times 2^\omega$ is analytic such that all sections $A_x$ are countable, then there is a perfect set which is free for $A$.

Theorem (Solecki–Spinas, 1999)  
If $A \subseteq \omega^\omega \times \omega^\omega$ is analytic such that all sections $A_x$ are $\sigma$-compact, then there is a superperfect set which is free for $A$. 
Definition

We say that a $\sigma$-ideal has the perfect set property if any Borel $I$-positive set contains a closed $I$-positive set.
Definition
We say that a $\sigma$-ideal has the *perfect set property* if any Borel $I$-positive set contains a closed $I$-positive set.

Remark
Note that if $P_I$ is $\omega^\omega$-bounding, then $I$ has the perfect set property.
Forcing idealized
Combinatorics of ideals
Closure ideals
Forcing connections
Borel degrees and Ramsey theorems

Notation

If $I$ is a $\sigma$-ideal, then by $I^{**}$ we denote the $\sigma$-ideal generated by compact sets in $I$. 
Notation

If $I$ is a $\sigma$-ideal, then by $I^{**}$ we denote the $\sigma$-ideal generated by compact sets in $I$.

Theorem (S.–Zapletal, 2010)

Suppose $I$ is $\Pi_1^1$ on $\Sigma_1^1$ $\sigma$-ideal on $X$ with the perfect set property and $P_I$ is proper. If $A \subseteq X \times X$ is analytic and the sections $A_x$ are in $I^{**}$, then there is an $I^{**}$-positive free set for $A$. 