Forcing, ideals and degrees of reals

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Joint work

This is joint work with Jindra Zapletal

Idealized forcings

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General question

A general question to ask is: what are the connections between descriptive set-theoretic properties of I and forcing properties of P_I?

For example: properness

The forcing P_I is proper if and only if for every countable $M \prec H_\kappa$ and $B \in P_I \cap M$ the set

$$\{x \in B : x \text{ is } P_I\text{-generic over } M\} \text{ is } I\text{-positive.}$$

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Theorem (Zapletal, 2002)

If I is generated by closed sets, then P_I is proper.

Generating by closed sets

We use the following notation. Given a σ-ideal I on a Polish space X we write I^* for the σ-ideal generated by closed sets which belong to I.

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General question

Another question is the following: are there any connections between forcing properties of P_I and P_{I^*}.

Relatives of the Miller forcing

Given an ideal J on ω (or just a hereditary family of subsets of ω) we consider the forcing notion $Q(J)$ of all subtrees $T \subseteq \omega^{<\omega}$ with the following property:

- for each $\tau \in T$ there is $\sigma \in T$ such that $\tau \subseteq \sigma$ and the set of immediate successors of σ in T is J-positive.

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If J is the family of finite subsets of ω, then $Q(J)$ is just the Miller forcing.

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Proposition (S.–Zapletal)

If J is a hereditary family of subsets of ω, then $Q(J)$ is forcing equivalent to P_{I_J} for some σ-ideal I_J generated by closed sets.

Katětov order

Given two hereditary families J and J' of subsets of $\mathrm{dom}(J)$ and $\mathrm{dom}(J')$, resectively we say that J is *Katětov below* J', in symbols $J \leq_K J'$ if there is a map

$$f : \mathrm{dom}(J') \to \mathrm{dom}(J)$$

such that $f^{-1}(a) \in J'$ for each $a \in J$.

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Notation

If J is a hereditary family of subsets of ω and $a \subseteq \omega$ is J-positive, then we write $J \restriction a$ for the family of subsets of a which are in J.

Nowhere dense

We write NWD for the ideal on $2^{<\omega}$ of those A such that for any $\tau \in 2^{<\omega}$ there is σ extending τ such that no further extension of σ falls into A.

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Theorem (S.–Zapletal, 2009)

The forcing $Q(J)$ does not add Cohen reals if and only if

$$J \restriction a \ \not\geq_K \ NWD$$

for any J-positive set a.

Fubini property

If $a \subseteq \mathrm{dom}(J)$ and $D \subseteq a \times 2^\omega$, then we write

$$\int_a D \; dJ = \{y \in 2^\omega : \{j \in a : \; \langle j, y \rangle \notin D\} \in J\}.$$

J has the *Fubini property* if for every real $\varepsilon > 0$, every J-positive set a and every Borel set $D \subseteq a \times 2^\omega$ with vertical sections of Lebesgue measure less than ε, the set $\int_a D \; dJ$ has outer measure at most ε.

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Definition

Let $0 < \varepsilon < 1$ be a real number. The ideal S_ε has as its domain all clopen subsets of 2^ω of Lebesgue measure less than ε, and it is generated by those sets a with $\bigcup a \neq 2^\omega$.

Remark

Obviously, the ideals S_ε as well as all families above them in the Katětov ordering fail to have the Fubini property.

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Theorem (Solecki, 2000)

Suppose F is an ideal on a countable set. Then either F has the Fubini property, or else for every (or equivalently, some) $\varepsilon > 0$ there is a F-positive set a such that

$$F \restriction a \geq_K S_\varepsilon.$$

Recall

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Theorem (S.–Zapletal, 2009)

Suppose that J is a universally measurable ideal. $Q(J)$ preserves outer Lebesgue measure if and only if J has the Fubini property.

Proof

Suppose on one hand that J fails to have the Fubini property. Find a sequence of J-positive sets $\langle b_n : n \in \omega \rangle$ such that

$$J \restriction b_n \geq_K S_{2^{-n}},$$

as witnessed by functions f_n. Consider the tree T of all sequences $t \in \mathrm{dom}(J)^{<\omega}$ such that $t(n) \in b_n$ for each $n \in \mathrm{dom}(t)$. Let $\dot{B}$ be a name for the set

$$\{z \in 2^\omega : \exists^\infty n \; z \in f_n(\dot{g}(n))\}.$$

T forces that the set $\dot{B}$ has measure zero, and the definition of the ideals S_ε shows that every ground model point in 2^ω is forced to belong to $\dot{B}$. Thus $Q(J)$ fails to preserve Lebesgue outer measure at least below the condition T.

Proof

On the other hand, suppose that the ideal J does have the Fubini property. Suppose that $Z \subseteq 2^\omega$ is a set of outer Lebesgue measure δ, $\dot{O}$ is a $Q(J)$-name for an open set of measure less or equal to $\varepsilon < \delta$, and $T \in Q(J)$ is a condition. We must find a point $z \in Z$ and a condition $S \leq T$ forcing $\check{z} \notin \dot{O}$.

By a standard fusion argument, thinning out the tree T if necessary, we may assume that there is a function $h : \mathrm{split}(T) \to \mathcal{O}$ such that

$$T \Vdash \dot{O} = \bigcup \{h(\dot{g} \restriction n+1) : \ \dot{g} \restriction n \in \mathrm{split}(T)\}.$$

Moreover, we can make sure that if $t_n \in T$ is the n-th splitting node, then $T \restriction t_n$ decides a subset of $\dot{O}$ with measure greater than $\varepsilon/2^n$. Hence, if we write $f(t_n) = \varepsilon/2^n$, then for every splitnode $t \in T$ and every $n \in \mathrm{succ}_T(t)$ we have $\mu(h(t^\frown n)) < f(t)$.

Proof

Now, for every splitnode $t \in T$ let

$$D_t = \{\langle O, x\rangle : x \in 2^\omega \ \wedge \ O \in \mathrm{succ}_T(t) \ \wedge \ x \in h(t^\frown O)\}.$$

It follows from universal measurability of J that the set $\int_{\mathrm{succ}_T(t)} D_t \; dJ$ is measurable. It has mass not greater than $f(t)$, by the Fubini assumption. Since $\sum_{t \in \mathrm{split}(T)} f(t) < \delta$, we can find

$$z \in Z \setminus \bigcup_{t \in \mathrm{split}(T)} \int_{\mathrm{succ}_T(t)} D_t \; dJ.$$

Let $S \subseteq T$ be the downward closure of those nodes $t^\frown n$ such that $t \in T$ is a splitnode and $n \in \mathrm{succ}_T(t)$ is such that $z \notin h(t^\frown n)$. S belongs to $Q(J)$ by the choice of the point z and $S \Vdash \check{z} \notin \dot{O}$, as required.

Definition

SPL is the family of nonsplitting subsets of $\omega^{<\omega}$, i.e. those $a \subseteq \omega^{<\omega}$ for which there is an infinite set $c \subseteq \omega$ such that $t \restriction c$ is constant for every $t \in a$.

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Theorem (S.–Zapletal, 2009)

Suppose that J is coanalytic hereditary, or an ideal with the Baire property. Then $Q(J)$ does not add splitting reals if and only if

$$J \restriction a \ \not\geq_K \ SPL$$

for every J-positive a.

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Question

Is SPL a Borel set?

Definition

We say that two reals x, y in a forcing extension *have the same continuous degree* if there is a **ground model partial homeomorphism** f of the real line (both domain and range should be G_δ such that

$$f(x) = y.$$

We say that a forcing *adds one continuous degree* if all reals in the extension have the same continuous degree.

Definition

We say that an ideal J has the *discrete set property* if it is not Katětov above the ideal generated by discrete subsets of the rationals.

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Proposition (S.–Zapletal)

If J has the discrete set property, then $Q(J)$ adds one continuous degree.

Definition

Fix a Polish space X and its countable basis $\mathcal{O}$ of open sets. For a set $a \subseteq \mathcal{O}$, define

$$\mathrm{cl}(a) = \{x \in X : \forall \varepsilon > 0\ \exists O \in a \quad O \subseteq B_\varepsilon(x)\},$$

where $B_\varepsilon(x)$ stands for the ball centered at x with radius ε. For a σ-ideal I on X we write

$$J_I = \{a \subseteq \mathcal{O} : \mathrm{cl}(a) \in I\}.$$

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$$J_I = \{a \subseteq \mathcal{O} : \mathrm{cl}(a) \in I\}.$$

Remark

Note that if X is compact and J_I is analytic, then it follows from the Kechris–Louveau–Woodin theorem that J_I is $F_{\sigma\delta}$

Definition

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Proposition (S.–Zapletal, 2009)

J_I is weakly selective for any σ-ideal I.

Theorem (Category Dichotomy, Hrušák, 2008)

If I is a Borel ideal, then

- either $I \leq_K NWD$,
- or $ED \leq_K I \restriction a$ for some I-positive a,

where ED is the ideal on $\omega \times \omega$ generated by graphs of functions and vertical sections.

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Corollary

If a Borel ideal J is weakly selective, then $I \leq_K NWD$.

Conjecture

If J is a dense (tall) $F_{\sigma\delta}$ weakly selective ideal on ω, then there exists a Polish space with a countable base $\mathcal{O}$ and a σ-ideal I on X such that under some identification of ω and $\mathcal{O}$ the ideal J becomes J_I.

Theorem (S.–Zapletal, 2009)

If I is a σ-ideal on a Polish space such that P_I is not equivalent to the Cohen forcing under any condition, then

$$P_{I^*} \equiv Q(J_I).$$

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Proposition (S.–Zapletal, 2009)

If I is such that P_{I^*} is equivalent to the Miller, Sacks or Cohen forcing, then $I = I^*$.

Theorem (S.–Zapletal, 2009)

If I is a σ-ideal such that P_I is proper and does not add Cohen reals, then $J_I \restriction a \not\geq_K NWD$ for any J_I-positive a.

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If I is a σ-ideal such that P_I is proper and does not add Cohen reals, then $J_I \restriction a \not\geq_K NWD$ for any J_I-positive a.

Corollary

If I is such that P_I is proper and does not add Cohen reals, then P_{I^*} inherites these properties.

Theorem (S.–Zapletal, 2009)

If I is a σ-ideal such that P_I is proper and preserves outer Lebesgue measure, then J_I has the Fubini property.

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Corollary

If I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ such that P_I is proper and preserves outer Lebesgue measure, then P_{I^*} inherits these properties.

Theorem (S.–Zapletal, 2009)

If I is such that P_I is proper and ω^ω-bounding, then J_I has the discrete set property.

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Corollary

If I is such that P_I is proper and ω^ω-bounding, then P_{I^*} adds one continuous degree.

Definition

We say that two reals x, y in a forcing extension *have the same Borel degree* if there is a **ground model Borel automorphism** f of the real line such that

$$f(x) = y.$$

We say that a forcing *adds one Borel degree* if all reals in the extension have the same Borel degree.

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Remarks

Recall that the Sacks and Miller forcing add one Borel (or even continuous) degree. On the other hand, the Cohen forcing adds "many" Borel degrees.

Theorem (S.–Zapletal, 2009)

If I is a σ-ideal generated by closed sets, then every real in the extension is either a ground model real, a Cohen real, or else has the same Borel degree as the generic real.

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If I is a σ-ideal generated by closed sets, then every real in the extension is either a ground model real, a Cohen real, or else has the same Borel degree as the generic real.

Remark on the proof

The proof uses a technique motivated by the Gandy–Harrington topology. Instead of recursively coded analytic sets, we use the Borel sets coded inside a countable model M as a topology base.

Theorem (Continuous reading of names, Zapletal, 2002)

If I is a σ-ideal generated by closed sets, then every Borel map defined on a Borel I-positive set can be restricted to a Borel I-positive set, on which it is continuous.

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Theorem (S.–Zapletal, 2009)

If I is a σ-ideal generated by closed sets, then the following are equivalent:

- P_I does not add Cohen reals,
- every Borel map defined on a Borel I-positive set can be restricted to a Borel I-positive set, on which it is either 1-1 or constant.

Question

Suppose I is generated by closed sets and P_I does not add Cohen reals. Does P_I necessarily add one **continuous** degree?

Theorem (Mycielski, ?)

If $A \subseteq 2^\omega \times 2^\omega$ is analytic such that all sections A_x are countable, then there is a perfect set which is free for A.

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Theorem (Solecki–Spinas, 1999)

If $A \subseteq \omega^\omega \times \omega^\omega$ is analytic such that all sections A_x are σ-compact, then there is a superperfect set which is free for A.

Definition

We say that a σ-ideal has the *perfect set property* if any Borel I-positive set contains a closed I-positive set.

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Remark

Note that if P_I is ω^ω-bounding, then I has the perfect set property.

Notation

If I is a σ-ideal, then by I^{**} we denote the σ-ideal generated by **compact** sets in I.

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Theorem (S.–Zapletal, 2010)

Suppose I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal on X with the perfect set property and P_I is proper. If $A \subseteq X \times X$ is analytic and the sections A_x are in I^{**}, then there is an I^{**}-positive free set for A.