Sequential properties of measures

Piotr Borodulin–Nadzieja (Wrocław)

Winterschool 2011, Hejnice

joint work with Omar Selim (Norwich)
Space of probability measures

Notation

- $K$ - a (Hausdorff) compact space;
- $\mathbb{N} = \{1, 2, \ldots\}$;
- $P(K)$ - space of probability Borel measures on $K$.

Weak* convergence

A sequence $(\mu_n)$ from $P(K)$ is weak* convergent to $\mu$ if

$$\int_K f \, d\mu_n \to \int_K f \, d\mu$$

for each continuous $f : K \to \mathbb{R}$.
Space of probability measures

Notation

- $K$ - a (Hausdorff) compact space;
- $\mathbb{N} = \{1, 2, \ldots\}$;
- $P(K)$ - space of probability Borel measures on $K$.

Weak* convergence

A sequence $(\mu_n)$ from $P(K)$ is weak* convergent to $\mu$ if

$$\int_K f \, d\mu_n \to \int_K f \, d\mu$$

for each continuous $f : K \to \mathbb{R}$.
Weak* convergence in 0-dim spaces

Weak* convergence

A sequence \((\mu_n)\) from \(P(K)\) is weak* convergent to \(\mu\) if

\[
\int_K f \, d\mu_n \to \int_K f \, d\mu
\]

for each continuous \(f : K \to \mathbb{R}\).

Remark

If \(K\) is zero–dimensional, then \(\mu_n\) converges weakly to \(\mu\) if and only if

\[
\mu_n(A) \to \mu(A)
\]

for every clopen subset \(A \subseteq K\).
Levels of complexity in $P(K)$

Sequential closures

- $h: K \to h[K] \subseteq P(K)$ defined by $h(x) = \delta_x$ is a homeomorphism;
- $S_0(K) = \text{conv}(\{\delta_x: x \in K\})$;
- let $S_1(K)$ be the weak$^*$–sequential closure of $S_0(K)$;
- generally: let $S_\alpha(K)$ be the weak$^*$–sequential closure of $\bigcup_{\beta < \alpha} S_\beta(K)$;
- $S(K) = S_{\omega_1}(K)$. 

Piotr Borodulin–Nadzieja (Wrocław) Sequential properties of measures
Sequential closures

- $h: K \to h[K] \subseteq P(K)$ defined by $h(x) = \delta_x$ is a homeomorphism;
- $S_0(K) = \text{conv}(\{\delta_x : x \in K\})$;
- let $S_1(K)$ be the weak*–sequential closure of $S_0(K)$;
- generally: let $S_\alpha(K)$ be the weak*–sequential closure of $\bigcup_{\beta < \alpha} S_\beta(K)$;
- $S(K) = S_{\omega_1}(K)$. 

Levels of complexity in $P(K)$
Levels of complexity in $P(K)$

### Sequential closures

- $h: K \rightarrow h[K] \subseteq P(K)$ defined by $h(x) = \delta_x$ is a homeomorphism;
- $S_0(K) = \text{conv}(\{\delta_x : x \in K\})$;
- let $S_1(K)$ be the weak*–sequential closure of $S_0(K)$;
- generally: let $S_\alpha(K)$ be the weak*–sequential closure of $\bigcup_{\beta < \alpha} S_\beta(K)$;
- $S(K) = S_{\omega_1}(K)$. 
Levels of complexity in $P(K)$

Sequential closures

- $h: K \to h[K] \subseteq P(K)$ defined by $h(x) = \delta_x$ is a homeomorphism;
- $S_0(K) = \text{conv}(\{\delta_x : x \in K\})$;
- let $S_1(K)$ be the weak*–sequential closure of $S_0(K)$;
- generally: let $S_\alpha(K)$ be the weak*–sequential closure of $\bigcup_{\beta < \alpha} S_\beta(K)$;
- $S(K) = S_{\omega_1}(K)$. 

Piotr Borodulin–Nadzieja (Wrocław) Sequential properties of measures
Remark

If \( \mu \in S(K) \), then it has a separable carrier, i.e. a closed set \( F \subseteq K \) with \( \mu(F) = 1 \) (not necessarily the support).

Corollary

Let \( \mathcal{A} = Bor([0,1])/Null \) be the measure algebra and let \( R \) be its Stone space. Then the standard measure \( \lambda \) on \( R \) is in \( P(R) \) but not in \( S(R) \).
Remark
If $\mu \in S(K)$, then it has a separable carrier, i.e. a closed set $F \subseteq K$ with $\mu(F) = 1$ (not necessarily the support).

Corollary
Let $\mathcal{A} = Bor([0, 1])/Null$ be the measure algebra and let $R$ be its Stone space. Then the standard measure $\lambda$ on $R$ is in $P(R)$ but not in $S(R)$. 
Fact

A measure $\mu$ is in $S_1(K)$ if and only if it has a uniformly distributed sequence.

Theorems

Many spaces $K$ have property: $P(K) = S_1(K)$. E.g.

- scattered spaces;
- metric spaces;
- $2^{\omega_1}$ [Losert, 79];
- $2^c$ [Fremlin, 00’s].
Uniform distribution

Fact
A measure $\mu$ is in $S_1(K)$ if and only if it has a uniformly distributed sequence.

Theorems
Many spaces $K$ have property: $P(K) = S_1(K)$. E.g.
- scattered spaces;
- metric spaces;
- $2^{\omega_1}$ [Losert, 79];
- $2^c$ [Fremlin, 00’s].
Theorem (Plebanek, PBN)
If $K$ is Koppelberg compact, then $P(K) = S(K)$.

Problem 1
Is there a space $K$ such that $S_1(K) \neq S(K)$?

Problem 2
Is there a space $K$ such that $S_1(K) \neq S(K) = P(K)$?
Theorem (Plebanek, PBN)
If $K$ is Koppelberg compact, then $P(K) = S(K)$.

Problem 1
Is there a space $K$ such that $S_1(K) \neq S(K)$?

Problem 2
Is there a space $K$ such that $S_1(K) \neq S(K) = P(K)$?
Preliminaries

Problems

Theorem (Plebanek, PBN)
If $K$ is Koppelberg compact, then $P(K) = S(K)$.

Problem 1
Is there a space $K$ such that $S_1(K) \neq S(K)$?

Problem 2
Is there a space $K$ such that $S_1(K) \neq S(K) = P(K)$?
Asymptotic density

Asymptotic density function
We say that $A \subseteq \mathbb{N}$ has a density if the limit

$$\lim_{n \to \infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n} = d(A)$$

exists.

Density and weak* convergence
If every element of a Boolean algebra $\mathcal{A} \subseteq P(\mathbb{N})$ has a density, then for $\mu$ defined on the Stone space $K$ of $\mathcal{A}$ by $\mu(\hat{A}) = d(A)$ for each $A \in \mathcal{A}$ we have

$$\mu(\hat{A}) = \lim_{n \to \infty} \frac{\delta_1(A) + \ldots + \delta_n(A)}{n}.$$
Asymptotic density

Asymptotic density function

We say that \( A \subseteq \mathbb{N} \) has a density if the limit

\[
\lim_{n \to \infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n} = d(A)
\]

exists.

Density and weak* convergence

If every element of a Boolean algebra \( \mathcal{A} \subseteq P(\mathbb{N}) \) has a density, then for \( \mu \) defined on the Stone space \( K \) of \( \mathcal{A} \) by \( \mu(\widehat{A}) = d(A) \) for each \( A \in \mathcal{A} \) we have

\[
\mu(\widehat{A}) = \lim_{n \to \infty} \frac{\delta_1(A) + \ldots + \delta_n(A)}{n}.
\]
Density and weak* convergence

If every element of a Boolean algebra $\mathcal{A} \subseteq P(\mathbb{N})$ has a density, then for $\mu$ defined on the Stone space $K$ of $\mathcal{A}$ by $\mu(\hat{A}) = d(A)$ for each $A \in \mathcal{A}$ we have

$$
\mu(\hat{A}) = \lim_{n \to \infty} \frac{\delta_1(A) + \ldots + \delta_n(A)}{n}.
$$

Corollary

$$
\mu \in S_1(\mathbb{N}) \subseteq S_1(K).
$$
Asymptotic density

Density and weak* convergence

If every element of a Boolean algebra $\mathcal{A} \subseteq P(\mathbb{N})$ has a density, then for $\mu$ defined on the Stone space $K$ of $\mathcal{A}$ by $\mu(\hat{A}) = d(A)$ for each $A \in \mathcal{A}$ we have

$$
\mu(\hat{A}) = \lim_{n \to \infty} \frac{\delta_1(A) + \ldots + \delta_n(A)}{n}.
$$

Corollary

$$
\mu \in S_1(\mathbb{N}) \subseteq S_1(K).
$$
Density and weak* convergence

If every element of a Boolean algebra $\mathcal{A} \subseteq P(\mathbb{N})$ has a density, then for $\mu$ defined on the Stone space $K$ of $\mathcal{A}$ by $\mu(\hat{A}) = d(A)$ for each $A \in \mathcal{A}$ we have

$$\mu(\hat{A}) = \lim_{n \to \infty} \frac{\delta_1(A) + \ldots + \delta_n(A)}{n}.$$ 

Corollary

$$\mu \in S_1(\mathbb{N}) \subseteq S_1(K).$$
Relative density

Fix a sequence \((B_n)_{n \in \mathbb{N}}\) of infinite and pairwise disjoint subsets of \(\mathbb{N}\) such that \(\bigcup_n B_n = \mathbb{N}\).

Let \(n \in \mathbb{N}\). Enumerate \(B_n = \{b_1 < b_2 < \ldots\}\). For \(A \subseteq B_n\) let

\[
d_n(A) = d(\{i : b_i \in A\}).
\]

Limit of densities

Let \(d'(A) = \lim_{n \to \infty} d_n(A)\) provided this limit exist. If each element \(A\) of a Boolean algebra \(\mathcal{A} \subseteq P(\mathbb{N})\) is such that \(d'(A)\) exists, then \(\mu \in P(\text{Stone}(\mathcal{A}))\) defined by

\[
\mu(\hat{A}) = d'(A)
\]

is in \(S_2(\mathbb{N})\)
Limit of densities

Relative density

Fix a sequence $(B_n)_{n \in \mathbb{N}}$ of infinite and pairwise disjoint subsets of \( \mathbb{N} \) such that \( \bigcup_n B_n = \mathbb{N} \).

Let \( n \in \mathbb{N} \). Enumerate \( B_n = \{ b_1 < b_2 < \ldots \} \). For \( A \subseteq B_n \) let

\[ d_n(A) = d(\{ i : b_i \in A \}). \]

Limit of densities

Let \( d'(A) = \lim_{n \to \infty} d_n(A) \) provided this limit exist. If each element \( A \) of a Boolean algebra \( \mathcal{A} \subseteq P(\mathbb{N}) \) is such that \( d'(A) \) exists, then \( \mu \in P(\text{Stone}(\mathcal{A})) \) defined by

\[ \mu(\hat{A}) = d'(A) \]

is in \( S_2(\mathbb{N}) \).
Limit of densities

Relative density

Fix a sequence \((B_n)_{n \in \mathbb{N}}\) of infinite and pairwise disjoint subsets of \(\mathbb{N}\) such that \(\bigcup_n B_n = \mathbb{N}\).

Let \(n \in \mathbb{N}\). Enumerate \(B_n = \{b_1 < b_2 < \ldots\}\). For \(A \subseteq B_n\) let

\[ d_n(A) = d(\{i : b_i \in A\}). \]

Limit of densities

Let \(d'(A) = \lim_{n \to \infty} d_n(A)\) provided this limit exist. If each element \(A\) of a Boolean algebra \(\mathcal{A} \subseteq P(\mathbb{N})\) is such that \(d'(A)\) exists, then \(\mu \in P(\text{Stone}(\mathcal{A}))\) defined by

\[ \mu(\hat{A}) = d'(A) \]

is in \(S_2(\mathbb{N})\).
The domain of measure

Definition

Let $\mathcal{F}$ be the filter of density 1 sets and let $\mathcal{C}$ be an isomorphic image (via $\varphi$) of the Cantor algebra $\text{alg}(2^{<\omega})$ such that

$$d(\varphi(\sigma)) = 1/2^{|\sigma|}$$

for each $\sigma \in 2^{<\omega}$.

Definition

For each $n \in \mathbb{N}$, $B_n = \{b_1 < b_2 < \ldots\}$ and $A \subseteq \mathbb{N}$ let

$$A^n = \{b_i: i \in A\}$$

$$\mathcal{F}^n = \{F^n: F \in \mathcal{F}\}$$

$$\mathcal{C}^n = \{C^n: C \in \mathcal{C}\}$$
**The domain of measure**

**Definition**

Let $\mathcal{F}$ be the filter of density 1 sets and let $\mathcal{C}$ be an isomorphic image (via $\varphi$) of the Cantor algebra $\text{alg}(2^{<\omega})$ such that

$$d(\varphi(\sigma)) = 1/2^{|\sigma|}$$

for each $\sigma \in 2^{<\omega}$.

**Definition**

For each $n \in \mathbb{N}$, $B_n = \{b_1 < b_2 < \ldots\}$ and $A \subseteq \mathbb{N}$ let

$$A^n = \{b_i : i \in A\}$$

$$\mathcal{F}^n = \{F^n : F \in \mathcal{F}\}$$

$$\mathcal{C}^n = \{C^n : C \in \mathcal{C}\}$$
First step

Definition

Let $\mathcal{B}_n$ be the Boolean algebra generated by $\mathcal{C}^n$ and $\mathcal{F}^n$, $n \in \mathbb{N}$.

Let $\mathcal{U}$ consist of sets $U \subseteq \mathbb{N}$ such that

- $U \cap B_n \in \mathcal{B}_n$ for each $n$;
- $\lim_{n \to \infty} d_n(U \cap B_n) = 1$.

Let $\mathbb{A}_0$ be the Boolean algebra generated by $\mathcal{U}$ (and $K_0$ - its Stone space).

Properties

- $\mathcal{U}$ is an ultrafilter on $\mathbb{A}_0$;
- $\mu = \delta U$;
- $\mu \in S_2(\mathbb{N})$;
- $\mu \notin S_1(K_0 \setminus \{\mathcal{U}\})$. 
First step

**Definition**

Let $\mathbb{B}_n$ be the Boolean algebra generated by $\mathcal{C}^n$ and $\mathcal{F}^n$, $n \in \mathbb{N}$. Let $\mathcal{U}$ consist of sets $U \subseteq \mathbb{N}$ such that

- $U \cap B_n \in \mathbb{B}_n$ for each $n$;
- $\lim_{n \to \infty} d_n(U \cap B_n) = 1$.

Let $\mathbb{A}_0$ be the Boolean algebra generated by $\mathcal{U}$ (and $K_0$ - its Stone space).

**Properties**

- $\mathcal{U}$ is an ultrafilter on $\mathbb{A}_0$;
- $\mu = \delta U$;
- $\mu \in S_2(\mathbb{N})$;
- $\mu \notin S_1(K_0 \setminus \{U\})$. 
First step

**Definition**

Let $B_n$ be the Boolean algebra generated by $C^n$ and $F^n$, $n \in \mathbb{N}$. Let $\mathcal{U}$ consist of sets $U \subseteq \mathbb{N}$ such that

- $U \cap B_n \in B_n$ for each $n$;
- $\lim_{n \to \infty} d_n(U \cap B_n) = 1$.

Let $A_0$ be the Boolean algebra generated by $\mathcal{U}$ (and $K_0$ - its Stone space).

**Properties**

- $\mathcal{U}$ is an ultrafilter on $A_0$;
- $\mu = \delta \mathcal{U}$;
- $\mu \in S_2(\mathbb{N})$;
- $\mu \notin S_1(K_0 \setminus \{\mathcal{U}\})$. 
First step

Definition
Let $B_n$ be the Boolean algebra generated by $C^n$ and $F^n$, $n \in \mathbb{N}$. Let $\mathcal{U}$ consist of sets $U \subseteq \mathbb{N}$ such that
- $U \cap B_n \in B_n$ for each $n$;
- $\lim_{n \to \infty} d_n(U \cap B_n) = 1$.

Let $A_0$ be the Boolean algebra generated by $\mathcal{U}$ (and $K_0$ - its Stone space).

Properties
- $\mathcal{U}$ is an ultrafilter on $A_0$;
  - $\mu = \delta U$;
  - $\mu \in S_2(\mathbb{N})$;
  - $\mu \notin S_1(K_0 \setminus \{\mathcal{U}\})$. 
First step

**Definition**

Let $\mathcal{B}_n$ be the Boolean algebra generated by $\mathcal{C}^n$ and $\mathcal{F}^n$, $n \in \mathbb{N}$. Let $\mathcal{U}$ consist of sets $U \subseteq \mathbb{N}$ such that

- $U \cap B_n \in \mathcal{B}_n$ for each $n$;
- $\lim_{n \to \infty} d_n(U \cap B_n) = 1$.

Let $\mathbb{A}_0$ be the Boolean algebra generated by $\mathcal{U}$ (and $K_0$ - its Stone space).

**Properties**

- $\mathcal{U}$ is an ultrafilter on $\mathbb{A}_0$;
- $\mu = \delta \mathcal{U}$;
- $\mu \in S_2(\mathbb{N})$;
- $\mu \notin S_1(K_0 \setminus \{\mathcal{U}\})$. 
First step

Definition
Let $B_n$ be the Boolean algebra generated by $C^n$ and $F^n$, $n \in \mathbb{N}$. Let $U$ consist of sets $U \subseteq \mathbb{N}$ such that
- $U \cap B_n \in B_n$ for each $n$;
- $\lim_{n \to \infty} d_n(U \cap B_n) = 1$.

Let $A_0$ be the Boolean algebra generated by $U$ (and $K_0$ - its Stone space).

Properties
- $U$ is an ultrafilter on $A_0$;
- $\mu = \delta_U$;
- $\mu \in S_2(\mathbb{N})$;
- $\mu \not\in S_1(K_0 \setminus \{U\})$. 
First step

Definition
Let $\mathcal{B}_n$ be the Boolean algebra generated by $\mathcal{C}^n$ and $\mathcal{F}^n$, $n \in \mathbb{N}$. Let $\mathcal{U}$ consist of sets $U \subseteq \mathbb{N}$ such that
- $U \cap B_n \in \mathcal{B}_n$ for each $n$;
- $\lim_{n \to \infty} d_n(U \cap B_n) = 1$.
Let $\mathcal{A}_0$ be the Boolean algebra generated by $\mathcal{U}$ (and $K_0$ - its Stone space).

Properties
- $\mathcal{U}$ is an ultrafilter on $\mathcal{A}_0$;
- $\mu = \delta_\mathcal{U}$;
- $\mu \in S_2(\mathbb{N})$;
- $\mu \notin S_1(K_0 \setminus \{\mathcal{U}\})$. 
Second step

Theorem (Fremlin)

There is a *monomorphism mod* $\mathcal{F}$

$$\psi: \mathcal{K} \rightarrow \text{Sets with density}$$

such that $d(\psi(R)) = \lambda(R)$ for each $R$.

Final step

Extend $\mathcal{A}_0$ to $\mathcal{A}$ by all sets of the form

$$\bigcup_{n} (\psi(R))^n$$

for every $R \in \mathcal{K} \setminus \{0, 1\}$. Let $K$ be its Stone space.
Theorem (Fremlin)

There is a monomorphism mod $\mathcal{F}$

$$\psi: \mathcal{R} \to \text{Sets with density}$$

such that $d(\psi(R)) = \lambda(R)$ for each $R$.

Final step

Extend $A_0$ to $A$ by all sets of the form

$$\bigcup_{n}(\psi(R))^n$$

for every $R \in \mathcal{R} \setminus \{0, 1\}$. Let $K$ be its Stone space.
Corollary

Let $D \subseteq K$ be the (closed) set generated by $\mathcal{U}$.

- $\mu \in S_2(\mathbb{N})$;
- $\mu \notin S_1(K \setminus D)$;
- $\mu \notin S_1(D)$;
- finally, $\mu \notin S_1(K)$.

Remark

In the same manner for every $\alpha < \omega_1$ we can produce a space $K$ and a measure $\mu$ such that $\mu \in S_\alpha(K) \setminus S_\beta(K)$ for each $\beta < \alpha$. 
Corollary

Let $D \subseteq K$ be the (closed) set generated by $\mathcal{U}$.

- $\mu \in S_2(\mathbb{N})$;
- $\mu \notin S_1(K \setminus D)$;
- $\mu \notin S_1(D)$;
- finally, $\mu \notin S_1(K)$.

Remark

In the same manner for every $\alpha < \omega_1$ we can produce a space $K$ and a measure $\mu$ such that $\mu \in S_\alpha(K) \setminus S_\beta(K)$ for each $\beta < \alpha$. 
Corollary

Let $D \subseteq K$ be the (closed) set generated by $\mathcal{U}$.

- $\mu \in S_2(\mathbb{N})$;
- $\mu \notin S_1(K \setminus D)$;
- $\mu \notin S_1(D)$;
- finally, $\mu \notin S_1(K)$.

Remark

In the same manner for every $\alpha < \omega_1$ we can produce a space $K$ and a measure $\mu$ such that $\mu \in S_{\alpha}(K) \setminus S_{\beta}(K)$ for each $\beta < \alpha$. 
Corollary

Let $D \subseteq K$ be the (closed) set generated by $\mathcal{U}$.

- $\mu \in S_2(\mathbb{N})$;
- $\mu \notin S_1(K \setminus D)$;
- $\mu \notin S_1(D)$;
- finally, $\mu \notin S_1(K)$.

Remark

In the same manner for every $\alpha < \omega_1$ we can produce a space $K$ and a measure $\mu$ such that $\mu \in S_\alpha(K) \setminus S_\beta(K)$ for each $\beta < \alpha$. 
Corollary

Let $D \subseteq K$ be the (closed) set generated by $\mathcal{U}$.

- $\mu \in S_2(\mathbb{N})$;
- $\mu \notin S_1(K \setminus D)$;
- $\mu \notin S_1(D)$;
- finally, $\mu \notin S_1(K)$.

Remark

In the same manner for every $\alpha < \omega_1$ we can produce a space $K$ and a measure $\mu$ such that $\mu \in S_\alpha(K) \setminus S_\beta(K)$ for each $\beta < \alpha$. 
Theorem (Plebanek)

Under CH there is a space $K$ such that

- there is $\mu \in S_2(K) \setminus S_1(K)$
- $S(K) = P(K)$. 
Thank you for your attention!

Slides and a preprint concerning the subject will be available on

http://www.math.uni.wroc.pl/~pboro