Sequential properties of measures

Piotr Borodulin–Nadzieja (Wrocław)

Winterschool 2011, Hejnice

joint work with Omar Selim (Norwich)

Space of probability measures

Notation

- K - a (Hausdorff) compact space;
- $\mathbb{N} = \{1, 2, \ldots\}$;
- $P(K)$ - space of probability Borel measures on K.

Weak* convergence

A sequence (μ_n) from $P(K)$ is weak* convergent to μ if

$$\int_K f\, d\mu_n \to \int_K f\, d\mu$$

for each continuous $f\colon K \to \mathbb{R}$.

Space of probability measures

Notation

- K - a (Hausdorff) compact space;
- $\mathbb{N} = \{1, 2, \ldots\}$;
- $P(K)$ - space of probability Borel measures on K.

Weak* convergence

A sequence (μ_n) from $P(K)$ is weak* convergent to μ if

$$\int_K f \, d\mu_n \to \int_K f \, d\mu$$

for each continuous $f : K \to \mathbb{R}$.

Weak* convergence in 0-dim spaces

Weak* convergence

A sequence (μ_n) from $P(K)$ is weak* convergent to μ if

$$\int_K f\, d\mu_n \to \int_K f\, d\mu$$

for each continuous $f\colon K \to \mathbb{R}$.

Remark

If K is zero–dimensional, then μ_n converges weakly to μ if and only if

$$\mu_n(A) \to \mu(A)$$

for every clopen subset $A \subseteq K$.

Levels of complexity in P(K)

Sequential closures

- $h\colon K \to h[K] \subseteq P(K)$ defined by $h(x) = \delta_x$ is a homeomorphism;
- $S_0(K) = conv(\{\delta_x\colon x \in K\})$;
- let $S_1(K)$ be the weak*–sequential closure of $S_0(K)$;
- generally: let $S_\alpha(K)$ be the weak*–sequential closure of $\bigcup_{\beta<\alpha} S_\beta(K)$;
- $S(K) = S_{\omega_1}(K)$.

Levels of complexity in P(K)

Sequential closures

- $h\colon K \to h[K] \subseteq P(K)$ defined by $h(x) = \delta_x$ is a homeomorphism;
- $S_0(K) = conv(\{\delta_x\colon x \in K\})$;
- let $S_1(K)$ be the weak*–sequential closure of $S_0(K)$;
- generally: let $S_\alpha(K)$ be the weak*–sequential closure of $\bigcup_{\beta<\alpha} S_\beta(K)$;
- $S(K) = S_{\omega_1}(K)$.

Levels of complexity in P(K)

Sequential closures

- $h\colon K \to h[K] \subseteq P(K)$ defined by $h(x) = \delta_x$ is a homeomorphism;
- $S_0(K) = conv(\{\delta_x\colon x \in K\})$;
- let $S_1(K)$ be the weak*–sequential closure of $S_0(K)$;
- generally: let $S_\alpha(K)$ be the weak*–sequential closure of $\bigcup_{\beta<\alpha} S_\beta(K)$;
- $S(K) = S_{\omega_1}(K)$.

Levels of complexity in P(K)

Sequential closures

- $h\colon K \to h[K] \subseteq P(K)$ defined by $h(x) = \delta_x$ is a homeomorphism;
- $S_0(K) = conv(\{\delta_x\colon x \in K\})$;
- let $S_1(K)$ be the weak*–sequential closure of $S_0(K)$;
- generally: let $S_\alpha(K)$ be the weak*–sequential closure of $\bigcup_{\beta<\alpha} S_\beta(K)$;
- $S(K) = S_{\omega_1}(K)$.

A measure outside the sequential closure

Remark

If $\mu \in S(K)$, then it has a separable carrier, i.e. a closed set $F \subseteq K$ with $\mu(F) = 1$ (not necessarily the support).

Corollary

Let $\mathfrak{R} = Bor([0,1])/Null$ be the measure algebra and let R be its Stone space. Then the standard measure λ on R is in $P(R)$ but not in $S(R)$.

A measure outside the sequential closure

Remark

If $\mu \in S(K)$, then it has a separable carrier, i.e. a closed set $F \subseteq K$ with $\mu(F) = 1$ (not necessarily the support).

Corollary

Let $\mathfrak{R} = Bor([0,1])/Null$ be the measure algebra and let R be its Stone space. Then the standard measure λ on R is in $P(R)$ but not in $S(R)$.

Uniform distribution

Fact

A measure μ is in $S_1(K)$ if and only if it has a uniformly distributed sequence.

Theorems

Many spaces K have property: $P(K) = S_1(K)$. E.g.

- scattered spaces;
- metric spaces;
- 2^{ω_1} [Losert, 79];
- $2^{\mathfrak{c}}$ [Fremlin, 00's].

Uniform distribution

Fact

A measure μ is in $S_1(K)$ if and only if it has a uniformly distributed sequence.

Theorems

Many spaces K have property: $P(K) = S_1(K)$. E.g.

- scattered spaces;
- metric spaces;
- 2^{ω_1} [Losert, 79];
- $2^{\mathfrak{c}}$ [Fremlin, 00's].

Problems

Theorem (Plebanek, PBN)

If K is Koppelberg compact, then $P(K) = S(K)$.

Problem 1

Is there a space K such that

$$S_1(K) \neq S(K)?$$

Problem 2

Is there a space K such that

$$S_1(K) \neq S(K) = P(K)?$$

Problems

Theorem (Plebanek, PBN)

If K is Koppelberg compact, then $P(K) = S(K)$.

Problem 1

Is there a space K such that

$$S_1(K) \neq S(K)?$$

Problem 2

Is there a space K such that

$$S_1(K) \neq S(K) = P(K)?$$

Problems

Theorem (Plebanek, PBN)

If K is Koppelberg compact, then $P(K) = S(K)$.

Problem 1

Is there a space K such that

$$S_1(K) \neq S(K)?$$

Problem 2

Is there a space K such that

$$S_1(K) \neq S(K) = P(K)?$$

Asymptotic density

Asymptotic density function

We say that $A \subseteq \mathbb{N}$ has a density if the limit

$$\lim_{n\to\infty} \frac{|A \cap \{1,2,\ldots,n\}|}{n} = d(A)$$

exists.

Density and weak* convergence

If every element of a Boolean algebra $\mathfrak{A} \subseteq P(\mathbb{N})$ has a density, then for μ defined on the Stone space K of $\mathfrak{A}$ by $\mu(\widehat{A}) = d(A)$ for each $A \in \mathfrak{A}$ we have

$$\mu(\widehat{A}) = \lim_{n\to\infty} \frac{\delta_1(A) + \ldots + \delta_n(A)}{n}.$$

Asymptotic density

Asymptotic density function

We say that $A \subseteq \mathbb{N}$ has a density if the limit

$$\lim_{n\to\infty} \frac{|A \cap \{1,2,\ldots,n\}|}{n} = d(A)$$

exists.

Density and weak* convergence

If every element of a Boolean algebra $\mathfrak{A} \subseteq P(\mathbb{N})$ has a density, then for μ defined on the Stone space K of $\mathfrak{A}$ by $\mu(\widehat{A}) = d(A)$ for each $A \in \mathfrak{A}$ we have

$$\mu(\widehat{A}) = \lim_{n\to\infty} \frac{\delta_1(A) + \ldots + \delta_n(A)}{n}.$$

Asymptotic density

Density and weak* convergence

If every element of a Boolean algebra $\mathfrak{A} \subseteq P(\mathbb{N})$ has a density, then for μ defined on the Stone space K of $\mathfrak{A}$ by $\mu(\widehat{A}) = d(A)$ for each $A \in \mathfrak{A}$ we have

$$\mu(\widehat{A}) = \lim_{n\to\infty} \frac{\delta_1(A) + \ldots + \delta_n(A)}{n}.$$

Corollary

$$\mu \in S_1(\mathbb{N}) \subseteq S_1(K).$$

Asymptotic density

Density and weak* convergence

If every element of a Boolean algebra $\mathfrak{A} \subseteq P(\mathbb{N})$ has a density, then for μ defined on the Stone space K of $\mathfrak{A}$ by $\mu(\widehat{A}) = d(A)$ for each $A \in \mathfrak{A}$ we have

$$\mu(\widehat{A}) = \lim_{n\to\infty} \frac{\delta_1(A) + \ldots + \delta_n(A)}{n}.$$

Corollary

$$\mu \in S_1(\mathbb{N}) \subseteq S_1(K).$$

Asymptotic density

Density and weak* convergence

If every element of a Boolean algebra $\mathfrak{A} \subseteq P(\mathbb{N})$ has a density, then for μ defined on the Stone space K of $\mathfrak{A}$ by $\mu(\widehat{A}) = d(A)$ for each $A \in \mathfrak{A}$ we have

$$\mu(\widehat{A}) = \lim_{n\to\infty} \frac{\delta_1(A) + \ldots + \delta_n(A)}{n}.$$

Corollary

$$\mu \in S_1(\mathbb{N}) \subseteq S_1(K).$$

Limit of densities

Relative density

Fix a sequence $(B_n)_{n\in\mathbb{N}}$ of infinite and pairwise disjoint subsets of $\mathbb{N}$ such that $\bigcup_n B_n = \mathbb{N}$.
Let $n \in \mathbb{N}$. Enumerate $B_n = \{b_1 < b_2 < \ldots\}$. For $A \subseteq B_n$ let

$$d_n(A) = d(\{i\colon b_i \in A\}).$$

Limit of densities

Let $d'(A) = \lim_{n\to\infty} d_n(A)$ provided this limit exist. If each element A of a Boolean algebra $\mathfrak{A} \subseteq P(\mathbb{N})$ is such that $d'(A)$ exists, then $\mu \in P(Stone(\mathfrak{A}))$ defined by

$$\mu(\widehat{A}) = d'(A)$$

is in $S_2(\mathbb{N})$

Limit of densities

Relative density

Fix a sequence $(B_n)_{n\in\mathbb{N}}$ of infinite and pairwise disjoint subsets of $\mathbb{N}$ such that $\bigcup_n B_n = \mathbb{N}$.
Let $n \in \mathbb{N}$. Enumerate $B_n = \{b_1 < b_2 < \dots\}$. For $A \subseteq B_n$ let

$$d_n(A) = d(\{i\colon b_i \in A\}).$$

Limit of densities

Let $d'(A) = \lim_{n\to\infty} d_n(A)$ provided this limit exist. If each element A of a Boolean algebra $\mathfrak{A} \subseteq P(\mathbb{N})$ is such that $d'(A)$ exists, then $\mu \in P(Stone(\mathfrak{A}))$ defined by

$$\mu(\widehat{A}) = d'(A)$$

is in $S_2(\mathbb{N})$

Limit of densities

Relative density

Fix a sequence $(B_n)_{n\in\mathbb{N}}$ of infinite and pairwise disjoint subsets of $\mathbb{N}$ such that $\bigcup_n B_n = \mathbb{N}$.
Let $n \in \mathbb{N}$. Enumerate $B_n = \{b_1 < b_2 < \ldots\}$. For $A \subseteq B_n$ let

$$d_n(A) = d(\{i\colon b_i \in A\}).$$

Limit of densities

Let $d'(A) = \lim_{n\to\infty} d_n(A)$ provided this limit exist. If each element A of a Boolean algebra $\mathfrak{A} \subseteq P(\mathbb{N})$ is such that $d'(A)$ exists, then $\mu \in P(Stone(\mathfrak{A}))$ defined by

$$\mu(\widehat{A}) = d'(A)$$

is in $S_2(\mathbb{N})$

The domain of measure

Definition

Let $\mathcal{F}$ be the filter of density 1 sets and let $\mathbb{C}$ be an isomorphic image (via φ) of the Cantor algebra $alg(2^{<\omega})$ such that

$$d(\varphi(\sigma)) = 1/2^{|\sigma|}$$

for each $\sigma \in 2^{<\omega}$.

Definition

For each $n \in \mathbb{N}$, $B_n = \{b_1 < b_2 < \ldots\}$ and $A \subseteq \mathbb{N}$ let

$$A^n = \{b_i : i \in A\}$$

$$\mathcal{F}^n = \{F^n : F \in \mathcal{F}\}$$

$$\mathbb{C}^n = \{C^n : C \in \mathbb{C}\}$$

The domain of measure

Definition

Let $\mathcal{F}$ be the filter of density 1 sets and let $\mathbb{C}$ be an isomorphic image (via φ) of the Cantor algebra $alg(2^{<\omega})$ such that

$$d(\varphi(\sigma)) = 1/2^{|\sigma|}$$

for each $\sigma \in 2^{<\omega}$.

Definition

For each $n \in \mathbb{N}$, $B_n = \{b_1 < b_2 < \ldots\}$ and $A \subseteq \mathbb{N}$ let

$$A^n = \{b_i \colon i \in A\}$$

$$\mathcal{F}^n = \{F^n \colon F \in \mathcal{F}\}$$

$$\mathbb{C}^n = \{C^n \colon C \in \mathbb{C}\}$$

First step

Definition

Let $\mathbb{B}_n$ be the Boolean algebra generated by $\mathbb{C}^n$ and $\mathcal{F}^n$, $n \in \mathbb{N}$.
Let $\mathcal{U}$ consist of sets $U \subseteq \mathbb{N}$ such that

- $U \cap B_n \in \mathbb{B}_n$ for each n;
- $\lim_{n\to\infty} d_n(U \cap B_n) = 1$.

Let $\mathbb{A}_0$ be the Boolean algebra generated by $\mathcal{U}$ (and K_0 - its Stone space).

Properties

- $\mathcal{U}$ is an ultrafilter on $\mathbb{A}_0$;
- $\mu = \delta_{\mathcal{U}}$;
- $\mu \in S_2(\mathbb{N})$;
- $\mu \notin S_1(K_0 \setminus \{\mathcal{U}\})$.

First step

Definition

Let $\mathbb{B}_n$ be the Boolean algebra generated by $\mathbb{C}^n$ and $\mathcal{F}^n$, $n \in \mathbb{N}$.
Let $\mathcal{U}$ consist of sets $U \subseteq \mathbb{N}$ such that

- $U \cap B_n \in \mathbb{B}_n$ for each n;
- $\lim_{n\to\infty} d_n(U \cap B_n) = 1$.

Let $\mathbb{A}_0$ be the Boolean algebra generated by $\mathcal{U}$ (and K_0 - its Stone space).

Properties

- $\mathcal{U}$ is an ultrafilter on $\mathbb{A}_0$;
- $\mu = \delta_{\mathcal{U}}$;
- $\mu \in S_2(\mathbb{N})$;
- $\mu \notin S_1(K_0 \setminus \{\mathcal{U}\})$.

First step

Definition

Let $\mathbb{B}_n$ be the Boolean algebra generated by $\mathbb{C}^n$ and $\mathcal{F}^n$, $n \in \mathbb{N}$.
Let $\mathcal{U}$ consist of sets $U \subseteq \mathbb{N}$ such that

- $U \cap B_n \in \mathbb{B}_n$ for each n;
- $\lim_{n\to\infty} d_n(U \cap B_n) = 1$.

Let $\mathbb{A}_0$ be the Boolean algebra generated by $\mathcal{U}$ (and K_0 - its Stone space).

Properties

- $\mathcal{U}$ is an ultrafilter on $\mathbb{A}_0$;
- $\mu = \delta_{\mathcal{U}}$;
- $\mu \in S_2(\mathbb{N})$;
- $\mu \notin S_1(K_0 \setminus \{\mathcal{U}\})$.

First step

Definition

Let $\mathbb{B}_n$ be the Boolean algebra generated by $\mathbb{C}^n$ and $\mathcal{F}^n$, $n \in \mathbb{N}$. Let $\mathcal{U}$ consist of sets $U \subseteq \mathbb{N}$ such that

- $U \cap B_n \in \mathbb{B}_n$ for each n;
- $\lim_{n\to\infty} d_n(U \cap B_n) = 1$.

Let $\mathbb{A}_0$ be the Boolean algebra generated by $\mathcal{U}$ (and K_0 - its Stone space).

Properties

- $\mathcal{U}$ is an ultrafilter on $\mathbb{A}_0$;
- $\mu = \delta_{\mathcal{U}}$;
- $\mu \in S_2(\mathbb{N})$;
- $\mu \notin S_1(K_0 \setminus \{\mathcal{U}\})$.

First step

Definition

Let $\mathbb{B}_n$ be the Boolean algebra generated by $\mathbb{C}^n$ and $\mathcal{F}^n$, $n \in \mathbb{N}$.
Let $\mathcal{U}$ consist of sets $U \subseteq \mathbb{N}$ such that

- $U \cap B_n \in \mathbb{B}_n$ for each n;
- $\lim_{n\to\infty} d_n(U \cap B_n) = 1$.

Let $\mathbb{A}_0$ be the Boolean algebra generated by $\mathcal{U}$ (and K_0 - its Stone space).

Properties

- $\mathcal{U}$ is an ultrafilter on $\mathbb{A}_0$;
- $\mu = \delta_{\mathcal{U}}$;
- $\mu \in S_2(\mathbb{N})$;
- $\mu \notin S_1(K_0 \setminus \{\mathcal{U}\})$.

First step

Definition

Let $\mathbb{B}_n$ be the Boolean algebra generated by $\mathbb{C}^n$ and $\mathcal{F}^n$, $n \in \mathbb{N}$.
Let $\mathcal{U}$ consist of sets $U \subseteq \mathbb{N}$ such that

- $U \cap B_n \in \mathbb{B}_n$ for each n;
- $\lim_{n\to\infty} d_n(U \cap B_n) = 1$.

Let $\mathbb{A}_0$ be the Boolean algebra generated by $\mathcal{U}$ (and K_0 - its Stone space).

Properties

- $\mathcal{U}$ is an ultrafilter on $\mathbb{A}_0$;
- $\mu = \delta_{\mathcal{U}}$;
- $\mu \in S_2(\mathbb{N})$;
- $\mu \notin S_1(K_0 \setminus \{\mathcal{U}\})$.

First step

Definition

Let $\mathbb{B}_n$ be the Boolean algebra generated by $\mathbb{C}^n$ and $\mathcal{F}^n$, $n \in \mathbb{N}$.
Let $\mathcal{U}$ consist of sets $U \subseteq \mathbb{N}$ such that

- $U \cap B_n \in \mathbb{B}_n$ for each n;
- $\lim_{n\to\infty} d_n(U \cap B_n) = 1$.

Let $\mathbb{A}_0$ be the Boolean algebra generated by $\mathcal{U}$ (and K_0 - its Stone space).

Properties

- $\mathcal{U}$ is an ultrafilter on $\mathbb{A}_0$;
- $\mu = \delta_{\mathcal{U}}$;
- $\mu \in S_2(\mathbb{N})$;
- $\mu \notin S_1(K_0 \setminus \{\mathcal{U}\})$.

Second step

Theorem (Fremlin)

There is a *monomorphism mod* $\mathcal{F}$

$$\psi \colon \mathfrak{R} \to \text{Sets with density}$$

such that $d(\psi(R)) = \lambda(R)$ for each R.

Final step

Extend $\mathbb{A}_0$ to $\mathbb{A}$ by all sets of the form

$$\bigcup_n (\psi(R))^n$$

for every $R \in \mathfrak{R} \setminus \{0,1\}$. Let K be its Stone space.

Second step

Theorem (Fremlin)

There is a *monomorphism mod* $\mathcal{F}$

$$\psi\colon \mathfrak{R} \to \text{Sets with density}$$

such that $d(\psi(R)) = \lambda(R)$ for each R.

Final step

Extend $\mathbb{A}_0$ to $\mathbb{A}$ by all sets of the form

$$\bigcup_n (\psi(R))^n$$

for every $R \in \mathfrak{R} \setminus \{0, 1\}$. Let K be its Stone space.

The result

Corollary

Let $D \subseteq K$ be the (closed) set generated by $\mathcal{U}$.

- $\mu \in S_2(\mathbb{N})$;
- $\mu \notin S_1(K \setminus D)$;
- $\mu \notin S_1(D)$;
- finally, $\mu \notin S_1(K)$.

Remark

In the same manner for every $\alpha < \omega_1$ we can produce a space K and a measure μ such that $\mu \in S_\alpha(K) \setminus S_\beta(K)$ for each $\beta < \alpha$.

The result

Corollary

Let $D \subseteq K$ be the (closed) set generated by $\mathcal{U}$.

- $\mu \in S_2(\mathbb{N})$;
- $\mu \notin S_1(K \setminus D)$;
- $\mu \notin S_1(D)$;
- finally, $\mu \notin S_1(K)$.

Remark

In the same manner for every $\alpha < \omega_1$ we can produce a space K and a measure μ such that $\mu \in S_\alpha(K) \setminus S_\beta(K)$ for each $\beta < \alpha$.

The result

Corollary

Let $D \subseteq K$ be the (closed) set generated by $\mathcal{U}$.

- $\mu \in S_2(\mathbb{N})$;
- $\mu \notin S_1(K \setminus D)$;
- $\mu \notin S_1(D)$;
- finally, $\mu \notin S_1(K)$.

Remark

In the same manner for every $\alpha < \omega_1$ we can produce a space K and a measure μ such that $\mu \in S_\alpha(K) \setminus S_\beta(K)$ for each $\beta < \alpha$.

The result

Corollary

Let $D \subseteq K$ be the (closed) set generated by $\mathcal{U}$.

- $\mu \in S_2(\mathbb{N})$;
- $\mu \notin S_1(K \setminus D)$;
- $\mu \notin S_1(D)$;
- finally, $\mu \notin S_1(K)$.

Remark

In the same manner for every $\alpha < \omega_1$ we can produce a space K and a measure μ such that $\mu \in S_\alpha(K) \setminus S_\beta(K)$ for each $\beta < \alpha$.

The result

Corollary

Let $D \subseteq K$ be the (closed) set generated by $\mathcal{U}$.

- $\mu \in S_2(\mathbb{N})$;
- $\mu \notin S_1(K \setminus D)$;
- $\mu \notin S_1(D)$;
- finally, $\mu \notin S_1(K)$.

Remark

In the same manner for every $\alpha < \omega_1$ we can produce a space K and a measure μ such that $\mu \in S_\alpha(K) \setminus S_\beta(K)$ for each $\beta < \alpha$.

Better example under CH

Theorem (Plebanek)

Under CH there is a space K such that

- there is $\mu \in S_2(K) \setminus S_1(K)$
- $S(K) = P(K)$.

The end

Thank you for your attention!

Slides and a preprint concerning the subject will be available on

http://www.math.uni.wroc.pl/~pborod