A Minimal Prikry-type Forcing for Singularizing a Measurable Cardinal

Karen Räsch
University of Münster

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This talk introduces a minimal Prikry-type forcing, i.e., it has the typical properties of Prikry-type forcings while every generic extension by it has no proper intermediate models.

There are lots of minimal forcings, like Sacks forcing or Laver forcing. Other forcings such as Cohen forcing are not minimal.

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This work was inspired by the following result

**Theorem (Gitik, Kanovei, Koepke, 2010)**

Let $V[G]$ be a generic extension by classical Prikry forcing. Then every intermediate model is a Prikry extension by this forcing and is generated by some subsequence of the associated Prikry sequence.

Moreover, the intermediate models of $V$ and $V[G]$ ordered by inclusion are isomorphic to $\mathcal{P}(\omega)/\text{finite}$ ordered by almost inclusion.
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Moreover, the intermediate models of $V$ and $V[G]$ ordered by inclusion are isomorphic to $\mathcal{P}(\omega)/\text{finite}$ ordered by almost inclusion.
For the rest of the talk let $\kappa$ be a measurable cardinal.

The classical Prikry forcing is equivalent to a Prikry tree forcing. Inspired by this, we define the partial order $P_\mathcal{U}$.

We obtain a standard Prikry lemma for $P_\mathcal{U}$, which makes it worthy of being called a Prikry-type forcing.

The minimality of $P_\mathcal{U}$ is a direct consequence of:

**Theorem (Koepke, Schlicht, R., 2010)**

Let $V[G]$ be a generic extension by $P_\mathcal{U}$ where $\mathcal{U}$ is sequence of pairwise distinct normal measures on $\kappa$.

Then for every $X \in V[G]$ either $X \in V$ or $X$ generates the whole generic extension, i.e., $V[X] = V[G]$. 
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Then for every $X \in V[G]$ either $X \in V$ or $X$ generates the whole generic extension, i.e., $V[X] = V[G]$. 
Think of $u, v \in [\kappa]^{< \omega}$ as strictly increasing sequences of ordinals. By $u \subseteq v$ we mean that $u$ is an initial segment of $v$. Concatenation is denoted by the symbol $\circlearrowleft$; the restriction of the domain by $\uparrow$.

A tree is a non-empty subset of $[\kappa]^{< \omega}$ which is closed under initial segments. $\text{Lev}_k(T)$ denotes the $k$-th level of $T$.

We denote the minimal inner model of ZFC containing the set $X \subseteq V$ and incorporating $V$ by $V[X]$.

We say $X$ is $V$-constructibly equivalent to $Y$, in short $X \equiv_V Y$, if $V[X] = V[Y]$. 

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Preliminaries

- Think of $u, v \in [\kappa]^{<\omega}$ as strictly increasing sequences of ordinals. By $u \preceq v$ we mean that $u$ is an initial segment of $v$. Concatenation is denoted by the symbol $\upharpoonright$; the restriction of the domain by $\upharpoonright$.

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We will not deal with inner models of ZFC but with sets of ordinals.

Reason: Every intermediate inner model $V \subseteq M \subseteq V[G]$ of ZFC is generated by a single set. Hence consider all sets of ordinals in $V[G]$ with the equivalence relation $\equiv_V$.

Fix a sequence $\mathcal{U} = \langle U_\alpha : \alpha < \kappa \rangle$ of pairwise distinct normal measures on $\kappa$.

The consistency strength is $\text{ZFC} + \text{“there exists a measurable cardinal”}$ by a theorem of Kunen and Paris.

For the minimality proof we will use a family $\langle A_\alpha : \alpha < \kappa \rangle$ of pairwise disjoint subsets of $\kappa$ such that $A_\alpha \in U_\alpha$. 
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**Definition of a $\mathcal{U}$-Tree**

A set $T \subseteq [\kappa]^{<\omega}$ is called a $\mathcal{U}$-tree with trunk $t$ if

- $\langle T, \sqsubseteq \rangle$ is a tree.
- $t \in T$ and for all $u \in T$ we have $u \sqsubseteq t$ or $t \sqsubseteq u$.
- For all $u \in T$ if $t \sqsubseteq u$ then

$$\text{Suc}_T(u) := \{ \xi < \kappa : u^\frown \langle \xi \rangle \in T \} \in U_{\text{max}}(u).$$
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Introduction
Tree Prikry Forcing for a Sequence of normal measures
The Minimality of $\mathbb{P}_\mathcal{U}$
Partition Properties of $\mathcal{U}$-Trees
Forcing with $\mathbb{P}_\mathcal{U}$

An Image of a $\mathcal{U}$-Tree

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A Minimal Prikry-type Forcing for Singularizing a Measurable Cardinal
Some Properties of $\mathcal{U}$-Trees

- Let $u \in T$, $u \trianglerighteq t$. Then
  \[ T \upharpoonright u := \{ v \in T : u \trianglelefteq v \lor v \trianglelefteq u \} \]
is a $\mathcal{U}$-tree with trunk $u$ and $\langle t, T \rangle \trianglerighteq \langle u, T \upharpoonright u \rangle$.

- The intersection of less than $\kappa$ many $\mathcal{U}$-trees all having the same trunk $t$ is again a $\mathcal{U}$-tree with trunk $t$. 

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The Partial Order $\mathbb{P}_U$

**Definition**

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Colorings of $\mathcal{U}$-Trees

Lemma (Colorings of $\mathcal{U}$-trees)

Let $T$ be a $\mathcal{U}$-tree and $c : T \to \lambda$ with $\lambda < \kappa$.

Then there is a $\mathcal{U}$-tree $\bar{T} \subseteq T$ with the same trunk homogeneous for $c$, i.e., every two elements of $c$ on the same level get the same color.

Proof.

Straightforward induction.
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For $u, v \in [\kappa]^{<\omega}$ enumerate $u \cup v$ strictly increasing as $\{ \xi_i : i < n \}$ and define $\text{type}(u, v) \in 3^n$ by

$$\text{type}(u, v)(i) = \begin{cases} 
0 & \text{if } \xi_i \in u \setminus v \\
1 & \text{if } \xi_i \in v \setminus u \\
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**Proof.**

First use the normality to define a diagonal intersection of $\mathcal{U}$-trees. The proof itself is a quite technical induction with lots of case distinctions.
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Classical Prikry tree forcing is $\mathbb{P}_U$ when all $U_\alpha$ equal the same $\kappa$-complete nonprincipal ultrafilter $U$ over $\kappa$.

Let $G$ be generic on $\mathbb{P}_U$. As usual

$$f_G := \bigcup \{ t : \exists T \langle t, T \rangle \in G \}$$

is an $\omega$-sequence cofinal in $\kappa$, called Prikry sequence.

$G$ consists of all $U$-trees of which $f_G$ is a branch, i.e., $f_G \equiv V G$.

$\langle \mathbb{P}_U, \leq \rangle$ satisfies the $\kappa^+$-cc.

$\langle \mathbb{P}_U, \leq^* \rangle$ is $\kappa$-closed.
 Classical Prikry tree forcing is $\mathbb{P}_\mathcal{U}$ when all $\mathcal{U}_\alpha$ equal the same $\kappa$-complete nonprincipal ultrafilter $\mathcal{U}$ over $\kappa$.

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Lemma (Prikry lemma)

Let $\langle t, T \rangle \in \mathcal{P}_U$ and $\varphi$ a statement in the forcing language. Then there is a direct extension $\langle s, S \rangle \in \mathcal{P}_U$ of $\langle t, T \rangle$ deciding $\varphi$. 
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Proof.

Follows easily from the lemma about colorings of \( \mathcal{U} \)-trees.
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The following theorem sums up what we achieved so far.

Theorem

Let \( G \) be a generic filter on \( \mathcal{P}_\mathcal{U} \). Then in \( V[G] \)

- \( \kappa \) is singular with \( \text{cf}(\kappa) = \aleph_0 \).
- No bounded subsets of \( \kappa \) are added.
- All cardinals are preserved and also all cofinalities but \( \kappa \)'s.
The Theorem

Remember:

- \( \mathcal{U} = \langle U_\alpha : \alpha < \kappa \rangle \) is a sequence of pairwise distinct normal measures on the measurable cardinal \( \kappa \).
- \( \langle A_\alpha : \alpha < \kappa \rangle \) is a family of pairwise disjoint subsets of \( \kappa \) such that \( A_\alpha \in U_\alpha \).

Theorem (Koepke, Schlicht, R., 2010)

Let \( V[G] \) be a generic extension by \( P_\mathcal{U} \).

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The Theorem

Remember:

- $\mathcal{U} = \langle U_\alpha : \alpha < \kappa \rangle$ is a sequence of pairwise distinct normal measures on the measurable cardinal $\kappa$.
- $\langle A_\alpha : \alpha < \kappa \rangle$ is a family of pairwise disjoint subsets of $\kappa$ such that $A_\alpha \in U_\alpha$.

Theorem (Koepke, Schlicht, R., 2010)

Let $V[G]$ be a generic extension by $P_\mathcal{U}$.
Then for every $X \in V[G]$ either $X \in V$ or $X \equiv_V f_G$. 
Theorem (Koepke, Schlicht, R., 2010)

Let $V[G]$ be a generic extension by $\mathbb{P}_\kappa$.

Then for every $X \in V[G]$ either $X \in V$ or $X \equiv_V f_G$.

The proof splits into two parts:

Part I. Subsets of $\kappa$ in $V[G]$

Part II. Arbitrary sets of ordinals in $V[G]$

Proof of the Theorem

Theorem (Koepke, Schlicht, R., 2010)

*Let $V[G]$ be a generic extension by $P_\mathcal{U}$. Then for every $X \in V[G]$ either $X \in V$ or $X \equiv_V f_G$."

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**Part I.** Subsets of $\kappa$ in $V[G]$  

**Part II.** Arbitrary sets of ordinals in $V[G]$
Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_U$.

Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

We will use the lemma about graphs on $\mathcal{U}$-trees
Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_\mathcal{U}$. Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

We will use the lemma about graphs on $\mathcal{U}$-trees.

Lemma (Graphs on $\mathcal{U}$-trees)

Let $T$ be a $\mathcal{U}$-tree and $c : T^2 \rightarrow \lambda$, $\lambda < \kappa$. Then there is a $\mathcal{U}$-tree $\bar{T} \subseteq T$ with the same trunk such that the value of $c$ only depends on the type of the arguments.
Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $P_{\mathcal{U}}$.

Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

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Lemma

Let $T$ be a $\mathcal{U}$-tree. Then there is $\bar{T} \subseteq T$ with the same trunk such that for all $u, v \in T$ with $u(n) \neq v(n)$, we have $u(m) \neq v(m)$ for all $m \geq n$ in both domains.

Proof.

Simply restrict $\text{Suc}_T(u)$ to $A_{\text{max}(u)}$ for all $u \in T$. \qed
Theorem (Part I)

Let $V[G]$ be a generic extension by $P_u$. Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

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Simply restrict $Succ_T(u)$ to $A_{max}(u)$ for all $u \in T$. 

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Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $P_u$. Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Sketch of the proof.

Let $\dot{X}$ be a name for some $X \subseteq \kappa$ and $\langle t, T \rangle \in P_u$.

Goal: Find $p \leq \langle t, T \rangle$ such that $p \Vdash (\dot{X} \in V \lor \dot{X} \equiv_V f)$.

By the Prikry lemma assume that for all $u \in T$ the condition $\langle u, T \upharpoonright u \rangle$ already decides $\dot{X}$ up to $\max(u)$.

For $u \in T$ define $X_u := \{ \xi < \max(u) : \langle u, T \upharpoonright u \rangle \Vdash \xi \in \dot{X} \}$.

Consider $c : T \times T \to 2$, where

$$\langle u, v \rangle \mapsto 1 \text{ iff } X_u \cap \max(v) = X_v \cap \max(u).$$
Proof of the Theorem – Part I

Theorem (Part I)

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Proof of the Theorem – Part I

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Consider $c : T \times T \rightarrow 2$, where

\[
\langle u, v \rangle \mapsto 1 \text{ iff } X_u \cap \max(v) = X_v \cap \max(u).
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### Proof of the Theorem – Part I

**Theorem (Part I)**

Let $V[G]$ be a generic extension by $\mathbb{P}_u$. Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

**Sketch of the proof.**

Let $\dot{X}$ be a name for some $X \subseteq \kappa$ and $\langle t, T \rangle \in \mathbb{P}_u$.

**Goal:** Find $p \leq \langle t, T \rangle$ such that $p \Vdash (\dot{X} \in V \lor \dot{X} \equiv_V f)$.

By the Prikry lemma assume that for all $u \in T$ the condition $\langle u, T \upharpoonright u \rangle$ already decides $\dot{X}$ up to $\max(u)$.

For $u \in T$ define $X_u := \{ \xi < \max(u) : \langle u, T \upharpoonright u \rangle \Vdash \dot{\xi} \in \dot{X} \}$.

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**Proof of the Theorem – Part I**

**Theorem (Part I)**

Let $V[G]$ be a generic extension by $P_U$.

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Let $\dot{X}$ be a name for some $X \subseteq \kappa$ and $\langle t, T \rangle \in P_U$.

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Proof of the Theorem – Part I

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Let $V[G]$ be a generic extension by $\mathbb{P}_u$. Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Sketch of the proof.

Let $\dot{X}$ be a name for some $X \subseteq \kappa$ and $\langle t, T \rangle \in \mathbb{P}_u$.

Goal: Find $p \leq \langle t, T \rangle$ such that $p \Vdash (\dot{X} \in V \vee \dot{X} \equiv_V \dot{f})$.

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$$\langle u, v \rangle \mapsto 1 \quad \text{iff} \quad X_u \cap \max(v) = X_v \cap \max(u).$$
**Theorem (Part I)**

Let $V[G]$ be a generic extension by $\mathbb{P}_\kappa$. Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

**Sketch of the proof (continued).**

Thin out $T$ and obtain $\bar{T} \subseteq T$ such that

- the values of $c$ on $\bar{T} \times \bar{T}$ only depend on the type
- for all $u, v \in \bar{T}$ with $u(n) \neq v(n)$, we have $u(m) \neq v(m)$ for all $m \geq n$ in both domains.

**Claim 1.** Let $s \in \bar{T}$ and $n < \omega$. Then $c$ is constant on the set

$\{\langle u, v \rangle \in \text{Lev}_{|s|+n}(\bar{T}^s) \times \text{Lev}_{|s|+n}(\bar{T}^s) : u(|s|) \neq v(|s|)\}$.

**Proof.** Later!
Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $P_u$.
Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

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Thin out $T$ and obtain $\overline{T} \subseteq T$ such that

- the values of $c$ on $\overline{T} \times \overline{T}$ only depend on the type
- for all $u, v \in \overline{T}$ with $u(n) \neq v(n)$, we have $u(m) \neq v(m)$ for all $m \geq n$ in both domains.

Claim 1. Let $s \in \overline{T}$ and $n < \omega$. Then $c$ is constant on the set

$$\{\langle u, v \rangle \in \text{Lev}_{|s|} + n (\overline{T} \upharpoonright s) \times \text{Lev}_{|s|} + n (\overline{T} \upharpoonright s) : u(|s|) \neq v(|s|)\}.$$

Proof. Later!
Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $P_u$. Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Sketch of the proof (continued).

Thin out $T$ and obtain $\bar{T} \subseteq T$ such that

- the values of $c$ on $\bar{T} \times \bar{T}$ only depend on the type
- for all $u, v \in \bar{T}$ with $u(n) \neq v(n)$, we have $u(m) \neq v(m)$ for all $m \geq n$ in both domains.

Claim 1. Let $s \in \bar{T}$ and $n < \omega$. Then $c$ is constant on the set

$$\{ \langle u, v \rangle \in \text{Lev}_{|s|+n}(\bar{T} \uparrow s) \times \text{Lev}_{|s|+n}(\bar{T} \uparrow s) : u(|s|) \neq v(|s|) \}.$$ 

Proof. Later!
Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_u$. Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Sketch of the proof (continued).

Thin out $T$ and obtain $\tilde{T} \subseteq T$ such that

- the values of $c$ on $\tilde{T} \times \tilde{T}$ only depend on the type
- for all $u, v \in \tilde{T}$ with $u(n) \neq v(n)$, we have $u(m) \neq v(m)$ for all $m \geq n$ in both domains.

Claim 1. Let $s \in \tilde{T}$ and $n < \omega$. Then $c$ is constant on the set

\[ \{\langle u, v \rangle \in \text{Lev}_{|s|+n}(\tilde{T} \upharpoonright s) \times \text{Lev}_{|s|+n}(\tilde{T} \upharpoonright s) : u(|s|) \neq v(|s|)\} \].

Proof. Later!
Proof of the Theorem – Part I

Theorem (Part I)

Let \( V[G] \) be a generic extension by \( P_\mu \).
Then for every \( X \subseteq \kappa \) in \( V[G] \) either \( X \in V \) or \( X \equiv_V f_G \).

Sketch of the proof (continued).

Claim 2. \( \langle t, \bar{T} \rangle \) forces \( \dot{X} \in V \lor \dot{X} \equiv_V \dot{f} \).

Proof.
How to construct \( f_G \) from \( \dot{X}^G \): Assume we know \( s := f_G \upharpoonright m \).

Case 1. There is \( n > 0 \) such that the only value of \( c \) on
\( \{ \langle u, v \rangle \in \text{Lev}_{m+n}(\bar{T} \upharpoonright s) \times \text{Lev}_{m+n}(\bar{T} \upharpoonright s) : u(m) \neq v(m) \} \) is 0.
Then all \( v \in \text{Lev}_{m+n}(\bar{T} \upharpoonright s) \) with \( v(m) \neq f_G(m) \) satisfy
\( c(v, f_G \upharpoonright (m + n)) = 0 \).
Hence \( X_v \neq \dot{X}^G \cap \max(v) \). This uniquely determines \( f_G(m) \).
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Proof of the Theorem – Part I

**Theorem (Part I)**

Let $V[G]$ be a generic extension by $P_{u}$. Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_{V} f_{G}$. 

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**Sketch of the proof (continued).**

**Claim 2.** $\langle t, \bar{T} \rangle$ forces $\dot{X} \in V \lor \dot{X} \equiv_{V} \dot{f}$.  

**Proof.** Let $G$ be generic, $\langle t, \bar{T} \rangle \in G$.

How to construct $f_{G}$ from $\dot{X}^{G}$: Assume we know $s := f_{G} \upharpoonright m$.

**Case 1.** There is $n > 0$ such that the only value of $c$ on $\{\langle u, v \rangle \in \text{Lev}_{m+n}(\bar{T} \upharpoonright s) \times \text{Lev}_{m+n}(\bar{T} \upharpoonright s) : u(m) \neq v(m)\}$ is 0. Then all $v \in \text{Lev}_{m+n}(\bar{T} \upharpoonright s)$ with $v(m) \neq f_{G}(m)$ satisfy $c(v, f_{G} \upharpoonright (m+n)) = 0$.

Hence $X_{v} \neq \dot{X}^{G} \cap \max(v)$. This uniquely determines $f_{G}(m)$. 

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Theorem (Part I)

Let $V[G]$ be a generic extension by $P_u$. Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Sketch of the proof (continued).

Claim 2. $\langle t, \bar{T} \rangle$ forces $\dot{X} \in V \lor \dot{X} \equiv_V f$.

Proof. We have $X_{f_G \uparrow (k+1)} = \dot{X}^G \cap f_G(k)$ for all $k$. Assume $\dot{X}^G \notin V$.

How to construct $f_G$ from $\dot{X}^G$: Assume we know $s := f_G \uparrow m$.

Case 1. There is $n > 0$ such that the only value of $c$ on 
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Then all $v \in \text{Lev}_{m+n}(\bar{T} \uparrow s)$ with $v(m) \neq f_G(m)$ satisfy $c(v, f_G \uparrow (m + n)) = 0$.

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Theorem (Part I)

Let \( V[G] \) be a generic extension by \( \mathbb{P}_\mu \).
Then for every \( X \subseteq \kappa \) in \( V[G] \) either \( X \in V \) or \( X \equiv_V f_G \).

Sketch of the proof (continued).

Claim 2. \( \langle t, \bar{T} \rangle \) forces \( \dot{X} \in V \lor \dot{X} \equiv_V f \).

Proof. We have \( X_{f_G \uparrow (k+1)} = \dot{X}^G \cap f_G(k) \) for all \( k \). Assume \( \dot{X}^G \notin V \).

How to construct \( f_G \) from \( \dot{X}^G \): Assume we know \( s := f_G \uparrow m \).

Case 1. There is \( n > 0 \) such that the only value of \( c \) on \( \{ \langle u, v \rangle \in \text{Lev}_{m+n}(\bar{T} \uparrow s) \times \text{Lev}_{m+n}(\bar{T} \uparrow s) : u(m) \neq v(m) \} \) is 0.

Then all \( v \in \text{Lev}_{m+n}(\bar{T} \uparrow s) \) with \( v(m) \neq f_G(m) \) satisfy \( c(v, f_G \uparrow (m+n)) = 0 \).
Hence \( X_v \neq \dot{X}^G \cap \max(v) \). This uniquely determines \( f_G(m) \).
Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_\mathcal{U}$.
Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Sketch of the proof (continued).

Claim 2. $\langle t, \bar{T} \rangle$ forces $\dot{X} \in V \lor \dot{X} \equiv_V f$.

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Case 1. There is $n > 0$ such that the only value of $c$ on
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**Theorem (Part I)**

Let $V[G]$ be a generic extension by $P_{\mathcal{U}}$. Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

**Sketch of the proof (continued).**

**Claim 2.** $\langle t, \bar{T} \rangle$ forces $\dot{X} \in V \lor \dot{X} \equiv_V \dot{f}$.

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**Case 1.** There is $n > 0$ such that the only value of $c$ on

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Proof of the Theorem – Part I

**Theorem (Part I)**

*Let $V[G]$ be a generic extension by $\mathbb{P}_u$. Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.***

**Sketch of the proof (continued).**

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**Proof.** We have $X_{f_G \uparrow (k+1)} = \dot{X}^G \cap f_G(k)$ for all $k$. Assume $\dot{X}^G \notin V$.  

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Hence $X_v \neq \dot{X}^G \cap \max(v)$. This uniquely determines $f_G(m)$.  

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Theorem (Part I)

Let $V[G]$ be a generic extension by $P_u$. Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

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Claim 2. $\langle t, \overline{T} \rangle$ forces $\dot{X} \in V \lor \dot{X} \equiv_V \dot{f}$.

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Then all $v \in \text{Lev}_{m+n}(\overline{T} \uparrow s)$ with $v(m) \neq f_G(m)$ satisfy $c(v, f_G \uparrow (m + n)) = 0$.

Hence $X_v \neq \dot{X}^G \cap \max(v)$. This uniquely determines $f_G(m)$.
Proof of the Theorem – Part I

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Then for every $X \subseteq \kappa$ in $V[G]$ either $X \subseteq V$ or $X \equiv V \ f_G$.

Sketch of the proof (continued).

Claim 2. $\langle t, \vec{T} \rangle$ forces $\dot{X} \subseteq V \lor \dot{X} \equiv V \ f$.

Proof. We have $X_{f_G \upharpoonright (k+1)} = \dot{X}^G \cap f_G(k)$ for all $k$. Assume $\dot{X}^G \notin V$.

How to construct $f_G$ from $\dot{X}^G$: Assume we know $s := f_G \upharpoonright m$.

Case 1. There is $n > 0$ such that the only value of $c$ on
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is $0$.

Then all $v \in \text{Lev}_{m+n}(\vec{T} \upharpoonright s)$ with $v(m) \neq f_G(m)$ satisfy
$c(v, f_G \upharpoonright (m + n)) = 0$.

Hence $X_v \neq \dot{X}^G \cap \max(v)$. This uniquely determines $f_G(m)$. 

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How to construct $f_G$ from $\dot{X}^G$: Assume we know $s := f_G \uparrow m$.

Case 1.

Case 2. For all $n > 0$ the only value of $c$ on 
$\{\langle u, v \rangle \in \text{Lev}_{m+n}(\bar{T} \uparrow s) \times \text{Lev}_{m+n}(\bar{T} \uparrow s) : u(m) \neq v(m)\}$ is 1.
Then $X_{s \cap \langle \xi \rangle} = \dot{X}^G \cap \xi$ for all $\xi$, i.e., $\dot{X}^G = \bigcup_{\xi \in \text{Suc}_{\bar{T}}(s)} X_{s \cap \langle \xi \rangle} \in V$. □
Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $P_{\mathcal{U}}$. Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Sketch of the proof (continued).

Claim 2. $\langle t, \bar{T} \rangle$ forces $\dot{X} \in V \lor \dot{X} \equiv_V \dot{f}$.

Proof. We have $X_{f_G \uparrow (k+1)} = \dot{X}^G \cap f_G(k)$ for all $k$. Assume $\dot{X}^G \notin V$.

How to construct $f_G$ from $\dot{X}^G$: Assume we know $s := f_G \uparrow m$.

Case 1.

Case 2. For all $n > 0$ the only value of $c$ on

$\{\langle u, v \rangle \in \text{Lev}_{m+n}(\bar{T} \uparrow s) \times \text{Lev}_{m+n}(\bar{T} \uparrow s) : u(m) \neq v(m)\}$

is 1.

Then $X_{s \uparrow \langle \xi \rangle} = \dot{X}^G \cap \xi$ for all $\xi$, i.e., $\dot{X}^G = \bigcup_{\xi \in \text{Suc}_{\bar{T}(s)}} X_{s \uparrow \langle \xi \rangle} \in V$. □
Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $P_u$.
Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Sketch of the proof (continued).

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Case 1. $\check{\ }$

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Then $X_{s \upharpoonright \langle \xi \rangle} = \dot{X}^G \cap \xi$ for all $\xi$, i.e., $\dot{X}^G = \bigcup_{\xi \in \text{Suc}_{\bar{T}}(s)} X_{s \upharpoonright \langle \xi \rangle} \in V$. $\square$
Theorem (Part I)

Let $V[G]$ be a generic extension by $P_\mathcal{U}$.

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Proof of the Theorem – Part I

Claim 1

Let $s \in \bar{T}$ and $n < \omega$. Then $c$ is constant on the set

$$\{ \langle u, v \rangle \in \text{Lev}_{|s|+n}(\bar{T} \upharpoonright s) \times \text{Lev}_{|s|+n}(\bar{T} \upharpoonright s) : u(|s|) \neq v(|s|) \}.$$  

Proof of Claim 1.

First remember that

- the values of $c$ on $\bar{T} \times \bar{T}$ only depend on the type,
- $u(m) \neq v(m)$ for all $\langle u, v \rangle$ in the above set, all $|s| \leq m < |s| + n$.

If there are $\langle u, v \rangle$ in the above set with $c(u, v) = 1$, then for all $\langle u', v' \rangle$ in the above set $c(u', v') = 1$.

Three steps to see this:

$$\text{type}(u, v) \leadsto t_{\text{alternating}} \leadsto t_{\text{successive}} \leadsto \text{type}(u', v').$$
Claim 1

Let \( s \in \bar{T} \) and \( n < \omega \). Then \( c \) is constant on the set

\[
\{ \langle u, v \rangle \in \text{Lev}_{|s|} \bar{T} \times \text{Lev}_{|s|} \bar{T} : u(|s|) \neq v(|s|) \}.
\]

Proof of Claim 1.

First remember that

- the values of \( c \) on \( \bar{T} \times \bar{T} \) only depend on the type,
- \( u(m) \neq v(m) \) for all \( \langle u, v \rangle \) in the above set, all \( |s| \leq m < |s| + n \).

If there are \( \langle u, v \rangle \) in the above set with \( c(u, v) = 1 \),
then for all \( \langle u', v' \rangle \) in the above set \( c(u', v') = 1 \).

Three steps to see this:

\[
\text{type}(u, v) \rightsquigarrow \mathbb{T}_{\text{alternating}} \rightsquigarrow \mathbb{T}_{\text{successive}} \rightsquigarrow \text{type}(u', v')
\]
Proof of the Theorem – Part I

Claim 1

Let $s \in \overline{T}$ and $n < \omega$. Then $c$ is constant on the set

$$\{ \langle u, v \rangle \in \text{Lev}_{|s|+n}(\overline{T} \upharpoonright s) \times \text{Lev}_{|s|+n}(\overline{T} \upharpoonright s) : u(|s|) \neq v(|s|) \}.$$ 

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Three steps to see this:

- type($u, v$) $\leadsto$ $\mathbb{T}_{\text{alternating}}$ $\leadsto$ $\mathbb{T}_{\text{successive}}$ $\leadsto$ type($u', v'$)
Proof of the Theorem – Part I

Claim 1

Let $s \in \overline{T}$ and $n < \omega$. Then $c$ is constant on the set
\[
\{ \langle u, v \rangle \in \text{Lev}_{|s|+n}(\overline{T}^{|s|}) \times \text{Lev}_{|s|+n}(\overline{T}^{|s|}) : u(|s|) \neq v(|s|) \}.
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Proof of Claim 1.

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- the values of $c$ on $\overline{T} \times \overline{T}$ only depend on the type,
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\[
\text{type}(u, v) \sim \mathbb{T}_{\text{alternating}} \sim \mathbb{T}_{\text{successive}} \sim \text{type}(u', v').
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Claim 1

Let $s \in \bar{T}$ and $n < \omega$. Then $c$ is constant on the set

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Three steps to see this:

$$\text{type}(u, v) \leadsto \text{t}_{\text{alternating}} \leadsto \text{t}_{\text{successive}} \leadsto \text{type}(u', v').$$
Proof of the Theorem – Part I

Claim 1

Let $s \in \bar{T}$ and $n < \omega$. Then $c$ is constant on the set

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Three steps to see this:

$$\text{type}(u, v) \sim t_{\text{alternating}} \sim t_{\text{successive}} \sim \text{type}(u', v').$$
Claim 1

Let $s \in \tilde{T}$ and $n < \omega$. Then $c$ is constant on the set

$$\{ \langle u, v \rangle \in \text{Lev}_{|s|+n}(\tilde{T} \upharpoonright s) \times \text{Lev}_{|s|+n}(\tilde{T} \upharpoonright s) : u(|s|) \neq v(|s|) \}.$$ 

Proof of Claim 1.

First remember that

- the values of $c$ on $\tilde{T} \times \tilde{T}$ only depend on the type,
- $u(m) \neq v(m)$ for all $\langle u, v \rangle$ in the above set, all $|s| \leq m < |s| + n$.

If there are $\langle u, v \rangle$ in the above set with $c(u, v) = 1$, then for all $\langle u', v' \rangle$ in the above set $c(u', v') = 1$.

Three steps to see this:

$$\text{type}(u, v) \leadsto \mathcal{t}_{\text{alternating}} \leadsto \mathcal{t}_{\text{successive}} \leadsto \text{type}(u', v')$$
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Let $s \in \bar{T}$ and $n < \omega$. Then $c$ is constant on the set

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Proof of Claim 1 (continued) for $n = 3$.

$$\langle 1, 0, 0, 1, 0, 1 \rangle \sim t_{\text{alternating}} \sim t_{\text{successive}} \sim \langle 1, 0, 0, 0, 1, 1 \rangle$$
Claim 1

Let $s \in \bar{T}$ and $n < \omega$. Then $c$ is constant on the set

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\]

Proof of Claim 1 (continued) for \( n = 3 \).

\[
\langle 1, 0, 0, 1, 0, 1 \rangle \sim t_{alternating} \sim t_{successive} \sim \langle 1, 0, 0, 0, 1, 1 \rangle
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Proof of the Theorem – Part I

Claim 1

Let $s \in \bar{T}$ and $n < \omega$. Then $c$ is constant on the set

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Proof of Claim 1 (continued) for $n = 3$.

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Proof of Claim 1 (continued) for \( n = 3 \).

\[\langle 1, 0, 0, 1, 0, 1 \rangle \sim t_{\text{alternating}} \sim t_{\text{successive}} \sim \langle 1, 0, 0, 0, 1, 1 \rangle\]
Theorem (Part II)

Let $V[G]$ be a generic extension by $\mathbb{P}_\mathcal{U}$. Then for every $X \in V[G]$ there exists $Y \subseteq \kappa$ in $V[G]$ with $X \equiv_V Y$.

Proof.

Proceed by induction on the least $\gamma$ with $X \subseteq \gamma$. Assume $\gamma > \kappa$.

Case 1. $\text{cf}(\gamma) \leq \kappa$.

We may assume that $\gamma$ is a limit ordinal. In $V[X]$ fix an increasing cofinal sequence $\langle \gamma_\xi : \xi < \text{cf}(\gamma) \rangle$ of ordinals in $\gamma$.

Fix a sequence $\langle Y_\xi : \xi < \text{cf}(\gamma) \rangle$ in $V[X]$ such that $Y_\xi \equiv_V X \cap \gamma_\xi$.

Since $\mathbb{P}_\mathcal{U}$ has the $\kappa^+$-cc we can code $\langle Y_\xi : \xi < \text{cf}(\gamma) \rangle$ and a recipe to obtain $X \cap \gamma_\xi$ from $Y_\xi$ in some $Y \subseteq \kappa$. 
Theorem (Part II)

Let $V[G]$ be a generic extension by $P_{\mathcal{U}}$. Then for every $X \in V[G]$ there exists $Y \subseteq \kappa$ in $V[G]$ with $X \equiv_V Y$.

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Let $V[G]$ be a generic extension by $\mathbb{P}_\mathcal{U}$.

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Fix a sequence $\langle Y_\xi : \xi < \text{cf}(\gamma) \rangle$ in $V[X]$ such that $Y_\xi \equiv V X \cap \gamma_\xi$.

Since $\mathbb{P}_\mathcal{U}$ has the $\kappa^+$-cc we can code $\langle Y_\xi : \xi < \text{cf}(\gamma) \rangle$ and a recipe to obtain $X \cap \gamma_\xi$ from $Y_\xi$ in some $Y \subseteq \kappa$. 
Theorem (Part II)

Let $V[G]$ be a generic extension by $P_\omega$. Then for every $X \in V[G]$ there exists $Y \subseteq \kappa$ in $V[G]$ with $X \equiv_V Y$.

Proof.

Proceed by induction on the least $\gamma$ with $X \subseteq \gamma$. Assume $\gamma > \kappa$.

Case 1. \(\text{cf}(\gamma) \leq \kappa\).

We may assume that $\gamma$ is a limit ordinal. In $V[X]$ fix an increasing cofinal sequence $\langle \gamma_\xi : \xi < \text{cf}(\gamma) \rangle$ of ordinals in $\gamma$.

Fix a sequence $\langle Y_\xi : \xi < \text{cf}(\gamma) \rangle$ in $V[X]$ such that $Y_\xi \equiv_V X \cap \gamma_\xi$. Since $P_\omega$ has the $\kappa^+$-cc we can code $\langle Y_\xi : \xi < \text{cf}(\gamma) \rangle$ and a recipe to obtain $X \cap \gamma_\xi$ from $Y_\xi$ in some $Y \subseteq \kappa$. 
Proof of the Theorem – Part II

Theorem (Part II)

Let $V[G]$ be a generic extension by $P_{\mathcal{U}}$. Then for every $X \in V[G]$ there exists $Y \subseteq \kappa$ in $V[G]$ with $X \equiv_V Y$.

Proof.

Proceed by induction on the least $\gamma$ with $X \subseteq \gamma$. Assume $\gamma > \kappa$.

Case 1. $\text{cf}(\gamma) \leq \kappa$.

We may assume that $\gamma$ is a limit ordinal. In $V[X]$ fix an increasing cofinal sequence $\langle \gamma_\xi : \xi < \text{cf}(\gamma) \rangle$ of ordinals in $\gamma$.

Fix a sequence $\langle Y_\xi : \xi < \text{cf}(\gamma) \rangle$ in $V[X]$ such that $Y_\xi \equiv_V X \cap \gamma_\xi$.

Since $P_{\mathcal{U}}$ has the $\kappa^+$-cc we can code $\langle Y_\xi : \xi < \text{cf}(\gamma) \rangle$ and a recipe to obtain $X \cap \gamma_\xi$ from $Y_\xi$ in some $Y \subseteq \kappa$. 

Karen Räsch

A Minimal Prikry-type Forcing for Singularizing a Measurable Cardinal
Theorem (Part II)

Let $V[G]$ be a generic extension by $\mathbb{P}_\mathcal{U}$.
Then for every $X \in V[G]$ there exists $Y \subseteq \kappa$ in $V[G]$ with $X \equiv_V Y$.

Proof.

Proceed by induction on the least $\gamma$ with $X \subseteq \gamma$. Assume $\gamma > \kappa$.

Case 1. $\text{cf}(\gamma) \leq \kappa$.

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Fix a sequence $\langle Y_\xi : \xi < \text{cf}(\gamma) \rangle$ in $V[X]$ such that $Y_\xi \equiv_V X \cap \gamma_\xi$.

Since $\mathbb{P}_\mathcal{U}$ has the $\kappa^+$-cc we can code $\langle Y_\xi : \xi < \text{cf}(\gamma) \rangle$ and a recipe to obtain $X \cap \gamma_\xi$ from $Y_\xi$ in some $Y \subseteq \kappa$. 
Proof of the Theorem – Part II

Theorem (Part II)

Let $V[G]$ be a generic extension by $\mathbb{P}_\mathcal{U}$.

Then for every $X \in V[G]$ there exists $Y \subseteq \kappa$ in $V[G]$ with $X \equiv V Y$.

Proof.

Proceed by induction on the least $\gamma$ with $X \subseteq \gamma$. Assume $\gamma > \kappa$.

Case 1. $\text{cf}(\gamma) \leq \kappa$. √

Case 2. $\text{cf}(\gamma) > \kappa$.

By the induction hypothesis either $X \cap \xi \in V$ or $X \cap \xi \equiv V f_G$ for every $\xi < \gamma$.

We may assume that $X \cap \xi \in V$ for all $\xi < \gamma$.

Show: If $X \cap \xi \in V$ for all $\xi < \gamma$, then $X \in V$. ∎
Theorem (Part II)

Let $V[G]$ be a generic extension by $\mathbb{P}_u$.

Then for every $X \in V[G]$ there exists $Y \subseteq \kappa$ in $V[G]$ with $X \equiv_V Y$.

Proof.

Proceed by induction on the least $\gamma$ with $X \subseteq \gamma$. Assume $\gamma > \kappa$.

Case 1. $\text{cf}(\gamma) \leq \kappa$.

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By the induction hypothesis either $X \cap \xi \in V$ or $X \cap \xi \equiv_V f_G$ for every $\xi < \gamma$.

We may assume that $X \cap \xi \in V$ for all $\xi < \gamma$.

Show: If $X \cap \xi \in V$ for all $\xi < \gamma$, then $X \in V$. 

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A Minimal Prikry-type Forcing for Singularizing a Measurable Cardinal
Proof of the Theorem – Part II

Theorem (Part II)

Let $V[G]$ be a generic extension by $\mathbb{P}_\kappa$.
Then for every $X \in V[G]$ there exists $Y \subseteq \kappa$ in $V[G]$ with $X \equiv V Y$.

Proof.

Proceed by induction on the least $\gamma$ with $X \subseteq \gamma$. Assume $\gamma > \kappa$.

Case 1. $\text{cf}(\gamma) \leq \kappa$. ✓

Case 2. $\text{cf}(\gamma) > \kappa$.
By the induction hypothesis either $X \cap \xi \in V$ or $X \cap \xi \equiv V f_G$ for every $\xi < \gamma$.

We may assume that $X \cap \xi \in V$ for all $\xi < \gamma$. Show: If $X \cap \xi \in V$ for all $\xi < \gamma$, then $X \in V$. 

Karen Räsch
A Minimal Prikry-type Forcing for Singularizing a Measurable Cardinal
Theorem (Part II)

Let $V[G]$ be a generic extension by $\mathbb{P}_\mathcal{U}$. Then for every $X \in V[G]$ there exists $Y \subseteq \kappa$ in $V[G]$ with $X \equiv_V Y$.

Proof.

Proceed by induction on the least $\gamma$ with $X \subseteq \gamma$. Assume $\gamma > \kappa$.

Case 1. $\text{cf}(\gamma) \leq \kappa$. √

Case 2. $\text{cf}(\gamma) > \kappa$.

By the induction hypothesis either $X \cap \xi \in V$ or $X \cap \xi \equiv_V f_G$ for every $\xi < \gamma$.

We may assume that $X \cap \xi \in V$ for all $\xi < \gamma$.

Show: If $X \cap \xi \in V$ for all $\xi < \gamma$, then $X \in V$. 
Theorem (Part II)

Let $V[G]$ be a generic extension by $\mathbb{P}_\mu$.

Then for every $X \in V[G]$ there exists $Y \subseteq \kappa$ in $V[G]$ with $X \equiv_Y Y$.

Proof.

Proceed by induction on the least $\gamma$ with $X \subseteq \gamma$. Assume $\gamma > \kappa$.

Case 1. $\text{cf}(\gamma) \leq \kappa$. ✓

Case 2. $\text{cf}(\gamma) > \kappa$.

By the induction hypothesis either $X \cap \xi \in V$ or $X \cap \xi \equiv_V f_G$ for every $\xi < \gamma$.

We may assume that $X \cap \xi \in V$ for all $\xi < \gamma$.

Show: If $X \cap \xi \in V$ for all $\xi < \gamma$, then $X \in V$. 

Karen Räsch

A Minimal Prikry-type Forcing for Singularizing a Measurable Cardinal
Now drop the assumption of normality. Then

- we still have a Prikry-type forcing,
- this forcing will not be minimal in general because of the Covering Theorem for $L[U]$,
- it is still possible to reduce the problem to subsets of $\kappa$. 
Now drop the assumption of normality. Then

- we still have a Prikry-type forcing,
- this forcing will not be minimal in general because of the Covering Theorem for $L[U]$,
- it is still possible to reduce the problem to subsets of $\kappa$. 
Now drop the assumption of normality. Then

- we still have a Prikry-type forcing,
- this forcing will not be minimal in general because of the Covering Theorem for $L[U]$,
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Thanks for listening! 😊