Sequence Selection Principles for Special Convergences

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Hausdorff - normal - perfectly normal topological spaces - $X$

- real valued functions
- a topological space of real valued continuous, Borel measurable functions with product topology denoted as $\mathcal{C}_p(X), \mathcal{B}_p(X)$, respectively

- discrete convergence of $\langle f_n : n \in \omega \rangle$:

  $$(\forall x \in X)(\exists n_0)(\forall n \in \omega)(n \geq n_0 \rightarrow f_n(x) = f(x))$$

- quasi-normal convergence:

  discrete $\rightarrow$ quasi-normal $\rightarrow$ pointwise
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Quasi-normal convergence of sequence $\langle f_n : n \in \omega \rangle$

**Pointwise convergence**
there exists $\langle \varepsilon_n : n \in \omega \rangle$ converging to 0 such that

$$(\forall m \in \omega)(\forall x \in X)(\exists n_0)(\forall n \in \omega)(n \geq n_0 \rightarrow |f_n(x) - f(x)| < \varepsilon_m)$$

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QN-property

$X$ has the property QN if each sequence of continuous functions converging to zero is converging to zero quasi-normally.

- $\mathfrak{b}$-Sierpiński set is a QN-set
- perfectly normal QN-space is a $\sigma$-space

wQN-property

$X$ has the property wQN if each sequence of continuous functions converging to zero has a subsequence converging to zero quasi-normally.

- $\text{QN} = \text{wQN}$ (Laver model), $\text{QN} \neq \text{wQN}$ (any model of $\text{ZFC} + t = \mathfrak{b}$)
- $\gamma$-space is a wQN-space
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$C_p(X)$ possesses the **sequence selection property**, shortly SSP, if for any functions $f, f_n, f_{n,m} : X \rightarrow \mathbb{R}$, $n, m \in \omega$, such that

a) $f_n \rightarrow f$ on $X$,

b) $f_{n,m} \rightarrow f_n$ on $X$ for every $n \in \omega$,

c) every $f, f_n, f_{n,m}$ is continuous,

there exists $\beta \in \omega \omega$ such that $f_{n,\beta(n)} \rightarrow f$ on $X$. 

$$\langle f_0, m : m \in \omega \rangle$$

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\langle f_1, m : m \in \omega \rangle \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \cdots \quad \longrightarrow \quad f_1 \\
\langle f_2, m : m \in \omega \rangle \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \cdots \quad \longrightarrow \quad f_2 \\
\langle f_3, m : m \in \omega \rangle \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \cdots \quad \longrightarrow \quad f_3 \\
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\[
\begin{align*}
\langle f_0, m : m \in \omega \rangle & \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \cdots \quad \to \quad f_0 \\
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\vdots & \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \to \quad f
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\langle f_1, m : m \in \omega \rangle & \quad \bullet \quad \circ \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \cdots \quad \rightarrow \quad f_1 \\
\langle f_2, m : m \in \omega \rangle & \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \cdots \quad \rightarrow \quad f_2 \\
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\[
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \ldots \quad \to \quad f
\]
Theorem (M. Scheepers, D. H. Fremlin)

Let $X$ be a topological space. Then the following are equivalent:

1. $X$ is a $wQN$-space;
2. $C_p(X)$ possesses SSP.

$X$ is Fréchet (or Fréchet–Urysohn) if for any $A \subseteq X$ and $x \in \overline{A}$ there is $x_n \in A$, $n \in \omega$ such that $x_n \rightarrow x$.

If $C_p(X)$ is Fréchet, then $C_p(X)$ possesses SSP.
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If $C_p(X)$ is Fréchet, then $C_p(X)$ possesses SSP.
X satisfies the pointwise–pointwise sequence selection principle, shortly PSP, if for any functions $f, f_n, f_{n,m} : X \rightarrow \mathbb{R}$, $n, m \in \omega$, such that

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\begin{align*}
\text{PSP} & \quad \text{QSP} & \quad \text{DSP} \\
\text{PSQ} & \quad \text{QSQ} & \quad \text{DSQ} \\
\text{PSD} & \quad \text{QSD} & \quad \text{DSD}
\end{align*}
$X$ satisfies the **quasi-normal–pointwise sequence selection principle**, shortly QSP, if for any functions $f, f_n, f_{n,m} : X \to \mathbb{R}$, $n, m \in \omega$, such that

a) $f_n \overset{\text{QN}}{\to} f$ on $X$,

b) $f_{n,m} \overset{\text{QN}}{\to} f_n$ on $X$ for every $n \in \omega$,

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$X$ satisfies the **pointwise–quasi-normal sequence selection principle**, shortly **PSQ**, if for any functions $f, f_n, f_{n,m} : X \rightarrow \mathbb{R}$, $n, m \in \omega$, such that

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<table>
<thead>
<tr>
<th>PSP</th>
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<th>DSP</th>
</tr>
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<tbody>
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</table>
Trivial relations

PSD $\xrightarrow{}$ QSD $\xrightarrow{}$ DSD

PSQ $\xrightarrow{}$ QSQ $\xrightarrow{}$ DSQ

PSP $\xrightarrow{}$ QSP $\xrightarrow{}$ DSP

PSD - a sequence of functions converging to zero would have to converge discretely.
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Trivial relations

\[ \text{QSD} \longrightarrow \text{DSD} \]
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DSD

PSQ → QSQ → DSQ

PSP → QSP → DSP

PSD - a sequence of functions converging to zero would have to converge discretely

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A perfectly normal topological space $X$

The following are equivalent:

1. $X$ satisfies PSQ;
2. $X$ satisfies QSQ;
3. $X$ satisfies DSD;
4. $X$ is a QN-space.
A perfectly normal topological space $X$

The following are equivalent:

1. $X$ satisfies $\text{PSQ}$;
2. $X$ satisfies $\text{QSQ}$;
3. $X$ satisfies $\text{DSD}$;
4. $X$ is a QN-space.

(DUPŠ Košice)
Covering properties

- **cover** \( U - \cup U = X \) and \( X \notin U \)

### \( S_1(\mathcal{A}, \mathcal{B}) \)-property

For each sequence \( \langle U_n : n \in \omega \rangle \) of covers from \( \mathcal{A} \), there exist sets \( U_n \in U_n \) such that \( \{ U_n : n \in \omega \} \in \mathcal{B} \).

### \( U_{\text{fin}}(\mathcal{A}, \mathcal{B}) \)-property

For each sequence \( \langle U_n : n \in \omega \rangle \) of covers from \( \mathcal{A} \) which do not contain a finite subcover, there exist finite subsets \( F_n \subseteq U_n \) such that \( \{ \cup F_n : n \in \omega \} \in \mathcal{B} \).

- **\( \gamma \)-cover** \( U - \) every \( x \in X \) lies in all but finitely many members of \( U \)
  - \( \gamma \) family of all countable open \( \gamma \)-covers \( \Gamma \)
Covering properties

- **cover** $\mathcal{U} - \bigcup \mathcal{U} = X$ and $X \notin \mathcal{U}$

### $S_1(\mathcal{A}, \mathcal{B})$-property

For each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of covers from $\mathcal{A}$, there exist sets $\mathcal{U}_n \in \mathcal{U}_n$ such that $\{U_n; n \in \omega\} \in \mathcal{B}$.

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- **$\gamma$-cover** $\mathcal{U} -$ every $x \in X$ lies in all but finitely many members of $\mathcal{U}$
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  - family of all countable open $\gamma$-covers: $\Gamma$
A perfectly normal topological space $X$

**σ-space** - every $F_\sigma$ subset of $X$ is a $G_\delta$ subset - $\Delta^0_2$

![Diagram]

1. **Laver model**: $QN = wQN$
2. **ZFC + $t = b$**: there is an $S_1(\Gamma, \Gamma)$-set which is not $\sigma$-space
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- $\sigma$-space
- $\Delta^0_2$
- $F_\sigma$
- $G_\delta$

QN $\equiv$ PSQ $\equiv$ QSQ $\equiv$ DSD

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Any QN-space satisfies the QSQ-principle.

A perfectly normal QN-space is a σ-space.

Let $X$ be a perfectly normal topological space. TFAE:
1. $X$ is a QN-space;
2. any Borel image of $X$ into $\omega^\omega$ is eventually bounded.
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Theorem (I. Reclaw)

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Theorem

*If a normal topological space $X$ satisfies QSQ, then $\text{Ind}(X) = 0$."

Assume that $X$ is a topological space with $\text{Ind}(X) = 0$:
- any $\Delta^0_2$-measurable function $f : X \rightarrow [0, 1]$ is a quasi-normal limit of a sequence of simple $\Delta^0_2$-measurable functions
- any simple $\Delta^0_2$-measurable function $g : X \rightarrow [0, 1]$ is a discrete limit of a sequence of simple continuous functions.

If $X$ is a perfectly normal topological space satisfying QSQ, then any Borel measurable function $f : X \rightarrow [0, 1]$ is a quasi-normal limit of a sequence of continuous functions.
**Theorem**

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Archangel’skiĭ’s properties \((\alpha_i)\)

For \(i = 1, 2, 3, 4\), a topological space \(Y\) is \((\alpha_i)\)-space if for any \(\langle S_n : n \in \omega \rangle\) of sequences converging to some point \(y \in Y\), there exists a sequence \(S\) converging to \(y\) such that:

\((\alpha_1)\) \(S_n \setminus S\) is infinite for all \(n \in \omega\);

\((\alpha_2)\) \(S_n \cap S\) is infinite for all \(n \in \omega\);

\((\alpha_3)\) \(S_n \cap S\) is infinite for infinitely many \(n \in \omega\);

\((\alpha_4)\) \(S_n \cap S \neq \emptyset\) for infinitely many \(n \in \omega\).

TFAE:

- \(X\) is a \(wQN\)-space;
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A perfectly normal topological space $X$

The following conditions are equivalent:

1. $X$ is a $QN$-space.
2. $C_p(X)$ possesses $(\alpha_1)$;
3. $B_p(X)$ possesses $(\alpha_1)$;
4. $B_p(X)$ possesses $(\alpha_2)$;
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7. $B_p(X)$ possesses SSP;
8. $X$ is a $QNB$-space;
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- family of all countable Borel covers / $\gamma$-covers: $\mathcal{B} / \mathcal{B}_\Gamma$
- family of all countable closed covers / $\gamma$-covers: $\mathcal{F} / \mathcal{F}_\Gamma$

The following conditions are equivalent:

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6. $X$ possesses the property $(\beta_1)$/Kočinac’s $\alpha(\Gamma, \Gamma)$;
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Thanks for your attention!