Sequence Selection Principles for Special Convergences

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- Hausdorff - normal - perfectly normal topological spaces - X
- real valued functions
- a topological space of real valued **continuous**, **Borel** measurable functions with product topology denoted as $C_p(X), \mathcal{B}_p(X)$, respectively
- discrete convergence of $\langle f_n : n \in \omega \rangle$:

$$(\forall x \in X)(\exists n_0)(\forall n \in \omega)(n \geq n_0 \to f_n(x) = f(x))$$

- quasi-normal convergence:
  - 1970’s - equal convergence - A. Császár and M. Laczkovich
  - 1990-2011 - quasi-normal convergence - Z. Bukovská, L. Bukovský, I. Recław, M. Repický, M. Scheepers, D. H. Fremlin, J. Haleš, M. Sakai, B. Tsaban, L. Zdomskyy

discrete $\to$ quasi-normal $\to$ pointwise

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discrete $\rightarrow$ quasi-normal $\rightarrow$ pointwise

Quasi-normal convergence of sequence $\langle f_n : n \in \omega \rangle$

Pointwise convergence

there exists $\langle \varepsilon_n : n \in \omega \rangle$ converging to 0 such that

$$(\forall m \in \omega)(\forall x \in X)(\exists n_0)(\forall n \in \omega)(n \geq n_0 \to |f_n(x) - f(x)| < \varepsilon_m)$$

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QN-property

X has the property QN if each sequence of continuous functions converging to zero is converging to zero quasi-normally.

- $\mathfrak{b}$-Sierpiński set is a QN-set
- perfectly normal QN-space is a σ-space

wQN-property

X has the property wQN if each sequence of continuous functions converging to zero has a subsequence converging to zero quasi-normally.

- QN$=$wQN (Laver model), QN$\neq$wQN (any model of $\mathbf{ZFC} + \mathfrak{t} = \mathfrak{b}$)
- γ-space is a wQN-space
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Selection principles

$C_p(X)$ possesses the **sequence selection property**, shortly **SSP**, if for any functions $f, f_n, f_{n,m} : X \longrightarrow \mathbb{R}$, $n, m \in \omega$, such that

a) $f_n \longrightarrow f$ on X,

b) $f_{n,m} \longrightarrow f_n$ on X for every $n \in \omega$,

c) every $f, f_n, f_{n,m}$ is continuous,

there exists $\beta \in {}^{\omega}\omega$ such that $f_{n,\beta(n)} \longrightarrow f$ on X.

$\langle f_{0,m} : m \in \omega \rangle$

$\langle f_{1,m} : m \in \omega \rangle$

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$$\begin{array}{lccccccccc}
\langle f_{0,m} : m \in \omega\rangle & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \cdots \\
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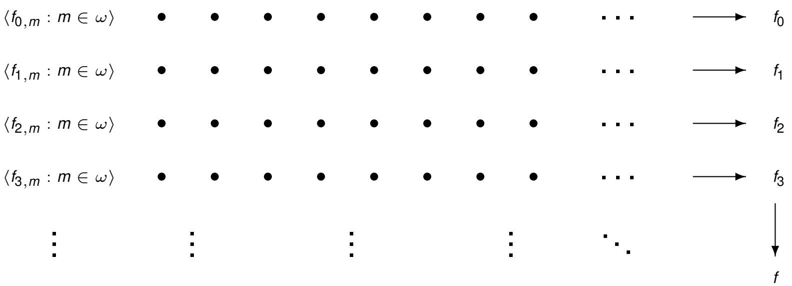

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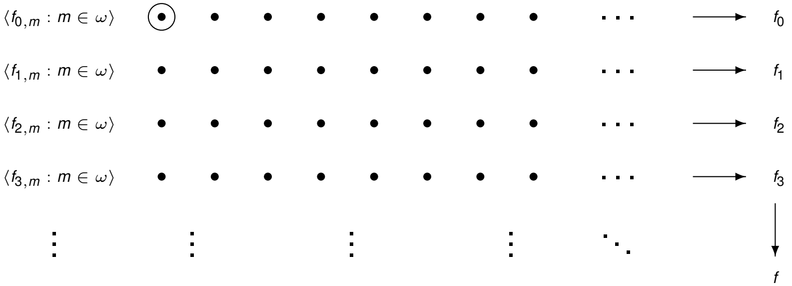

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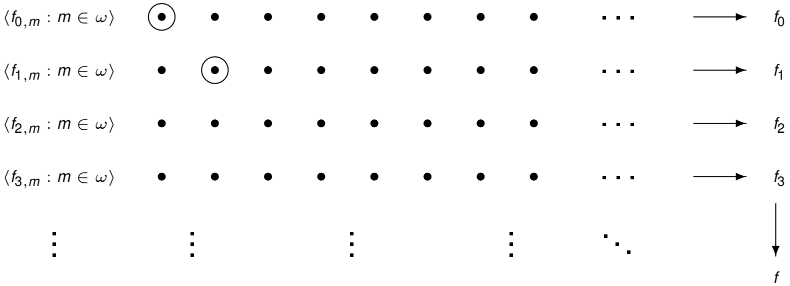

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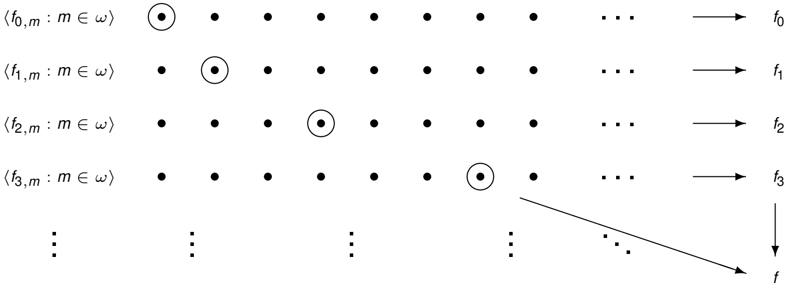

Theorem (M. Scheepers, D. H. Fremlin)

Let X be a topological space. Then the following are equivalent:

1. *X is a* wQN*-space;*
2. $C_p(X)$ *possesses* SSP.

X is **Fréchet** (or Fréchet–Urysohn) if for any $A \subseteq X$ and $x \in \overline{A}$ there is $x_n \in A, n \in \omega$ such that $x_n \longrightarrow x$.

If $C_p(X)$ is Fréchet, then $C_p(X)$ possesses SSP.

Theorem (M. Scheepers, D. H. Fremlin)

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If $\mathrm{C}_p(X)$ is Fréchet, then $\mathrm{C}_p(X)$ possesses SSP.

X satisfies the **pointwise–pointwise sequence selection principle**, shortly **PSP**, if for any functions $f, f_n, f_{n,m} : X \longrightarrow \mathbb{R}$, $n, m \in \omega$, such that

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PSP	QSP	DSP
PSQ	QSQ	DSQ
PSD	QSD	DSD

X satisfies the **quasi-normal–pointwise sequence selection principle**, shortly **QSP**, if for any functions $f, f_n, f_{n,m} : X \longrightarrow \mathbb{R}$, $n, m \in \omega$, such that

a) $f_n \xrightarrow{\text{QN}} f$ on X,

b) $f_{n,m} \xrightarrow{\text{QN}} f_n$ on X for every $n \in \omega$,

c) every $f_{n,m}$ is continuous,

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PSP	**QSP**	DSP
PSQ	QSQ	DSQ
PSD	QSD	DSD

X satisfies the **pointwise–quasi-normal sequence selection principle**, shortly **PSQ**, if for any functions $f, f_n, f_{n,m} : X \longrightarrow \mathbb{R}$, $n, m \in \omega$, such that

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PSP	QSP	DSP
PSQ	QSQ	DSQ
PSD	QSD	DSD

Trivial relations

$$\begin{array}{ccccc}
\mathrm{PSD} & \longrightarrow & \mathrm{QSD} & \longrightarrow & \mathrm{DSD} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{PSQ} & \longrightarrow & \mathrm{QSQ} & \longrightarrow & \mathrm{DSQ} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{PSP} & \longrightarrow & \mathrm{QSP} & \longrightarrow & \mathrm{DSP}
\end{array}$$

PSD - a sequence of functions converging to zero would have to converge discretely

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 & & \downarrow & & \downarrow \\
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\mathrm{PSP} & \longrightarrow & \mathrm{QSP} & \longrightarrow & \mathrm{DSP}
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PSD - a sequence of functions converging to zero would have to converge discretely

QSD - a sequence of functions converging to zero quasi-normally would have to converge discretely

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PSD - a sequence of functions converging to zero would have to converge discretely

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PSD - a sequence of functions converging to zero would have to converge discretely

QSD - a sequence of functions converging to zero quasi-normally would have to converge discretely

A perfectly normal topological space X

$$
\begin{array}{ccccc}
 & & & & \mathbf{DSD} \\
 & & & & \downarrow \\
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\downarrow & & \downarrow & & \downarrow \\
\mathrm{PSP} & \longrightarrow & \mathrm{QSP} & \longrightarrow & \mathrm{DSP}
\end{array}
$$

The following are equivalent:

1. X satisfies PSQ;
2. X satisfies QSQ;
3. X satisfies DSD;
4. X is a QN-space.

A perfectly normal topological space X

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\begin{array}{ccccc}
 & & & & \mathbf{DSD} \\
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\mathbf{PSQ} & \longrightarrow & \mathbf{QSQ} & \longrightarrow & \mathrm{DSQ} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{PSP} & \longrightarrow & \mathrm{QSP} & \longrightarrow & \mathrm{DSP}
\end{array}
$$

The following are equivalent:

1. X satisfies PSQ;
2. X satisfies QSQ;
3. X satisfies DSD;
4. X is a QN-space.

Covering properties

- **cover** $\mathcal{U}$ - $\cup\,\mathcal{U} = X$ and $X \notin \mathcal{U}$

$\mathrm{S}_1(\mathcal{A}, \mathcal{B})$-property

For each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of covers from $\mathcal{A}$, there exist sets $U_n \in \mathcal{U}_n$ such that $\{U_n;\ n \in \omega\} \in \mathcal{B}$.

$\mathrm{U}_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$-property

For each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of covers from $\mathcal{A}$ which do not contain a finite subcover, there exist finite subsets $\mathcal{F}_n \subseteq \mathcal{U}_n$ such that $\{\cup\,\mathcal{F}_n;\ n \in \omega\} \in \mathcal{B}$.

- γ **-cover** $\mathcal{U}$ - every $x \in X$ lies in all but finitely many members of $\mathcal{U}$
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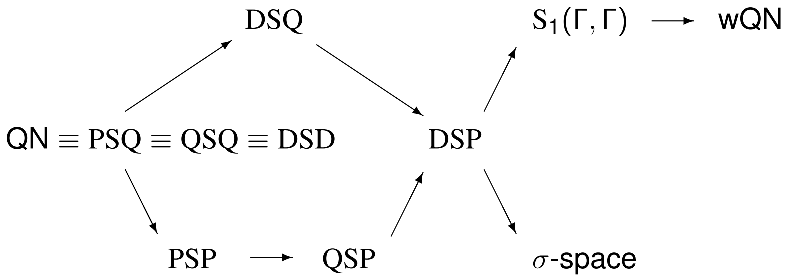
A perfectly normal topological space X

σ-space - every F_σ subset of X is a G_δ subset - $\mathbf{\Delta}^0_2$



1. **Laver model:** QN = wQN
2. $\mathrm{ZFC} + \mathfrak{t} = \mathfrak{b}$: there is an $S_1(\Gamma,\Gamma)$-set which is not σ-space

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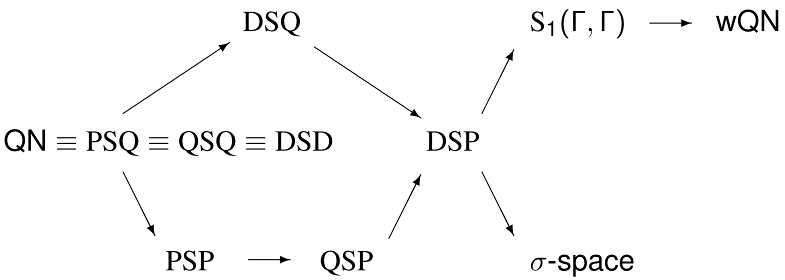

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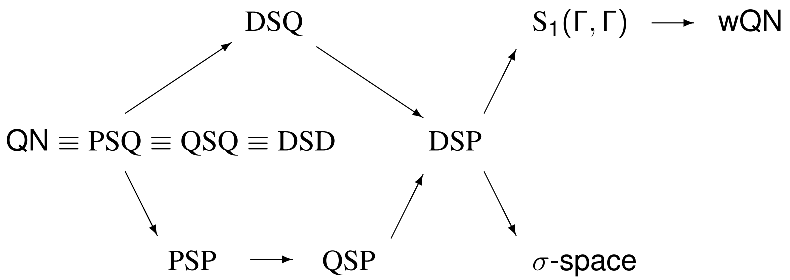

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Theorem

Any QN-space satisfies the QSQ-principle.

Theorem (I. Reclaw)

A perfectly normal QN-space is a σ-space.

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Let X be a perfectly normal topological space. TFAE:

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Approximation

Theorem

If a normal topological space X satisfies QSQ*, then* $\mathrm{Ind}(X) = 0$.

Assume that X is a topological space with $\mathrm{Ind}(X) = 0$:

- any Δ^0_2-measurable function $f : X \longrightarrow [0, 1]$ is a quasi-normal limit of a sequence of simple Δ^0_2-measurable functions
- any simple Δ^0_2-measurable function $g : X \longrightarrow [0, 1]$ is a discrete limit of a sequence of simple continuous functions

Theorem

If X is a perfectly normal topological space satisfying QSQ*, then any Borel measurable function $f : X \longrightarrow [0, 1]$ is a quasi-normal limit of a sequence of continuous functions.*

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Archangel'skiĭ's properties (α_i)

For $i = 1, 2, 3, 4$, a topological space Y is (α_i)-space if for any $\langle S_n : n \in \omega \rangle$ of sequences converging to some point $y \in Y$, there exists a sequence S converging to y such that:

- (α_1) $S_n \setminus S$ is infinite for all $n \in \omega$;
- (α_2) $S_n \cap S$ is infinite for all $n \in \omega$;
- (α_3) $S_n \cap S$ is infinite for infinitely many $n \in \omega$;
- (α_4) $S_n \cap S \neq \emptyset$ for infinitely many $n \in \omega$.

TFAE:

1. X is a wQN-space;
2. $C_p(X)$ possesses (α_2);
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A perfectly normal topological space X

The following conditions are equivalent:

1. X is a QN-space.
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6. $\mathcal{B}_p(X)$ possesses (α_4);
7. $\mathcal{B}_p(X)$ possesses SSP;
8. X is a $\mathrm{QN}\mathcal{B}$-space;
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The following conditions are equivalent:

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3. $\mathrm{Borel}(X)$ is weakly distributive / X possesses $\mathrm{U}_{\mathrm{fin}}(\mathcal{B},\mathcal{B}_\Gamma)$;
4. X possesses the property $S_1(\mathcal{F}_\Gamma,\mathcal{F}_\Gamma)$;
5. X possesses the property $S_1(\mathcal{B}_\Gamma,\mathcal{B}_\Gamma)$;
6. X possesses the property (β_1)/Kočinac's $\alpha_1(\Gamma,\Gamma)$;
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Thanks for your attention!