Universally Measurable Sets II

Paul Larson

Department of Mathematics
Miami University
Oxford, Ohio 45056

larsonpb@miamioh.edu

February 3, 2026

At the end last time we produced a nonmeager universally null set (so a universally measurable set without the Baire property) from the assumption $\mathfrak{d} = \mathrm{cof}(\mathcal{M}) \leq \mathsf{non}(\mathcal{N})$.

We will soon see that in the random model every universally null set is universally meager.

First we will show that $\mathsf{add}(\mathcal{N}) = \mathsf{max}\{\mathfrak{b}, \mathfrak{s}\}$ also implies that there is a universally measurable set without the Baire property.

$\mathbb{G}_0$ graphs

For $D \subseteq 2^{<\omega}$, let $G(D)$ be the graph on 2^ω connecting x and y if they disagree in exactly one place, and their longest common initial segment is in D.

A $\mathbb{G}_0$-graph is a graph of the form $G(D)$ when D is dense and has exactly one member of length n for each $n \in \omega$. We let $\mathbb{G}_0$ be the set of $\mathbb{G}_0$-graphs.

The $\mathbb{G}_0$-dichotomy (Solecki-Kechris-Todorceivic) says that if $G \in \mathbb{G}_0$ and H is an analytic graph on a Polish space, then either H has a Borel $\mathbb{N}$-coloring or there is a homomorphism from G to H.

More on $\mathbb{G}_0$-graphs

Each $G \in \mathbb{G}_0$ is acyclic and locally countable.

The connected components are the $\mathbb{E}_0$-degrees.

Comeagerly many $x \in 2^\omega$ lie in a connected component of G whose members all have infinite degree.

Measure 1 many $x \in 2^\omega$ have finite degree in G.

Baire colorings

Given a graph G on a Polish space X and a collection of sets Γ, let $\chi(G,\Gamma)$ be the smallest cardinality of a set of G-independent sets from Γ whose union contains X.

If $G \in \mathbb{G}_0$ and $A \subseteq 2^\omega$ is G-independent, then for no $s \in 2^{<\omega}$ is

$$A \cap [s]$$

comeager in s.

Letting Δ_{B} be the collection of subsets of 2^ω with the Baire property, it follows that

$$\chi(G, \Delta_{\mathrm{B}}) \geq \mathrm{cov}(\mathcal{M}).$$

Measurable colorings

Let Δ_{U} be the set of universally measurable subsets of 2^ω.

Ben Miller proved that $\mathrm{add}(\mathcal{N}) = \mathfrak{c}$ implies that, for any $G \in \mathbb{G}_0$,

$$\chi(G, \Delta_{\mathrm{U}}) \leq 3.$$

(Gaspar-Larson) The argument goes through assuming

$$\mathrm{add}(\mathcal{N}) = \max\{\mathfrak{b}, \mathfrak{s}\}.$$

Recall that the collection of universally measurable sets is closed under unions of cardinality less than $\mathrm{add}(\mathcal{N})$.

It is an open question whether ZFC implies that $\chi(G, \Delta_{\mathrm{U}}) \leq 3$ for any $G \in \mathbb{G}_0$.

Free functions

A (Borel) function $f \colon X \to X$ is said to be *free* if, for all $x \in X$ and $m \in \mathbb{N}$,

$$f^{(m)}(x) \neq x.$$

The graph $\mathcal{G}_f$ connects each $x \in X$ with $f(x)$.

Theorem. $\chi(\mathcal{G}_f, \Delta_{\mathrm{U}}) \leq 3$ if and only if there exists a universally measurable set $A \subseteq X$ such that for every $x \in X$ there exist $i, j \in \omega$ such that $f^i(x) \in A$ and $f^j(x) \notin A$.

Given $x \in X$, we call $\{f^i(x) : i \in \omega\}$ the *f-forward image* of x.

The proof of of Miller's theorem involves 3-coloring part of the graph G in a Borel way, and then proving the following lemma, which implies that $\chi(\mathcal{G}_f, \Delta_{\mathrm{U}}) \leq 3$.

Lemma. (B. Miller) Suppose that $\mathrm{add}(\mathcal{N}) = \mathfrak{c}$. Let X be a Polish space, $f : X \to X$ be a free Borel functon, and

$$\langle F_n : n \in \omega \rangle$$

be a $\subseteq$-increasing sequence of Borel equivalence relations whose union contains $\mathcal{G}_f$, such that no F_n-class contains any set of the form

$$\{f^i(x) : i \in \omega\}.$$

Then there exists a universally measurable set $A \subseteq X$ such that, for all $x \in X$ there exist $i, j \in \omega$ such that $f^i(x) \in A$ and $f^j(x) \notin A$.

We will sketch a proof of the lemma from the weaker hypothesis $\mathrm{add}(\mathcal{N}) = \max\{\mathfrak{b}, \mathfrak{s}\}$.

Given x, let $e(x)$ be the least n such that $xF_n f(x)$. For every x,

$$\{e(f^i(x)) : i \in \omega\}$$

is infinite.

It follows that for each Borel probability measure μ there is a function $g_\mu\colon \omega \to \omega$ such that, for each $n \in \omega$,

$$\mu(\{x : \exists i\, e(f^i(x)) \in (n, g_\mu(n)]\}) > 1 - (1/n).$$

Given $A \subseteq \omega$, let D_A be the set of x in X such that

$$\{i : e(f^i(x)) \in A\}$$

and

$$\{i : e(f^i(x)) \notin A\}$$

are both infinite.

Each D_A is Borel and f-invariant.

Lemma. Let μ be a Borel measure on 2^ω and let $A \subseteq \omega$ be such that A contains infinitely many intervals of the form $(n, g_\mu(n)]$, and is also disjoint from infinitely many such intervals. Then $\mu(D_A) = 1$.

$\max\{\mathfrak{b}, \mathfrak{s}\}$

Let γ be the least cardinality of a set $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that, for every increasing $f : \omega \to \omega$ there exists an $A \in \mathcal{A}$ such that the sets

$$\{n \in \omega : (n, f(n)] \subseteq A\}$$

and

$$\{n \in \omega : (n, f(n)] \cap A = \emptyset\}$$

are both infinite.

Then

$$\gamma = \max\{\mathfrak{b}, \mathfrak{s}\},$$

where $\mathfrak{s}$, the *splitting number*, is the smallest cardinality of a set $\mathcal{B} \subseteq \mathcal{P}(\omega)$ such that, for all infinite $A \subseteq \omega$ there exists $B \in \mathcal{B}$ with $A \cap B$ and $A \setminus B$ infinite.

Fix a family $\mathcal{A} = \langle A_\alpha : \alpha < \gamma \rangle$ witnessing the value of γ.

Each D_{A_α} is Borel, and thus universally measurable.

If $\gamma = \mathrm{add}(\mathrm{null})$, then

$$\bigcup_{\beta<\gamma} (D_{A_\beta} \setminus \bigcup_{\alpha<\beta} D_{A_\alpha})$$

is a universally measurable set as desired.

$\mathfrak{c} = \aleph_2$?

$\mathfrak{s} \leq \text{non}(\mathcal{M}), \mathfrak{d}, \text{non}(\mathcal{N})$.

If $\mathfrak{c} = \aleph_2$ and every universally measurable set has the Baire property, then

- $\aleph_2 = \max\{\mathfrak{b}, \mathfrak{s}\} > \text{add}(\mathcal{N}) = \aleph_1$ (by Miller-Gaspar-Larson) so
- $\aleph_2 = \mathfrak{d} = \text{non}(\mathcal{M}) = \mathrm{cof}(\mathcal{M})$ so
- $\aleph_1 = \text{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{M})$ (by Brendle-Larson) so
- $\mathfrak{s} = \aleph_1$ and $\mathfrak{b} = \aleph_2$.

This collection of values holds in the Laver and random/Laver models.

Codes

For the rest of this talk we will look at universally measurable sets in generic extensions.

Recall that a Polish space (X, τ) has a countable dense set and is complete with respect to some metric. We can treat the restriction of the metric to the dense set as a code for the space, and use this code to reinterpret the Polish space in generic extensions.

Similarly, each Borel (analytic/projective) subset of X has a definition (using a hereditarily countable parameter) which we can use to reinterpret the set in generic extensions.

Cohen and random (I)

Given an ideal I on a Polish space (X, τ), let P_I be the partial order $\mathrm{Borel}(X)/I$ (without the class for the emptyset), ordered by containment.

- Cohen forcing is P_I for I the set of meager subsets of 2^ω; it is also the partial order $2^{<\omega}$. Cohen forcing adds an element of 2^ω which is not in any (reinterpreted) meager Borel set from the ground model.
- Random forcing is P_I for I the set of Lebesgue-null subsets of 2^ω. Random forcing adds an element of 2^ω which is not in any Lebesgue-null Borel set from the ground model.

Cohen and random (II)

The universally null and universally meager sets can each consistently be contained in the other.

Starting with a model of GCH,

- A finite support iteration of Cohen forcing of length ω_2 forces that

$$\mathrm{UMeag}(2^\omega) \subseteq [2^\omega]^{\leq\aleph_1} \subseteq \mathrm{UNull}(2^\omega).$$

- A finite support iteration of random forcing of length ω_2 forces that

$$\mathrm{UNull}(2^\omega) \subseteq [2^\omega]^{\leq\aleph_1} \subseteq \mathrm{UMeag}(2^\omega).$$

We will sketch the proof over the next few slides.

Proof (I)

Suppose that $V \models$ GCH, and that G is V-generic for a finite-support iteration of Cohen or random forcing of length ω_2.

For each $\beta < \omega_2^V$, let G_β be the restriction of G to the first β stages of the forcing.

Proof (II)

For the second inclusions, note that if A is a set of reals of cardinality at most $\aleph_1$ in $V[G]$, then there is a $\beta < \omega_2^V$ such that $A \subseteq V[G_\beta]$.

Cohen forcing makes the set of ground model reals Lebesgue null, so (in the Cohen iteration case) A is Lebesgue null in $V[G_{\beta+1}]$.

Random forcing makes the set of ground model reals meager, so in the random iteration case A is meager in $V[G_{\beta+1}]$.

Recall that if every set of reals of cardinality at most $\aleph_1$ is Lebesgue null, then every set of reals of cardinality at most $\aleph_1$ is universally null (and similarly for meagerness).

Proof (III)

For the first inclusions, since universal nullness and meagerness of A are Π_2 properties over $(H(\aleph_1), \in, A)$:

If $A \subseteq P$ in $V[G]$ is universally meager (or null), then there is a $\beta < \omega_2^V$ such that $A \cap V[G_\beta]$ is universally meager (or null) in $V[G_\beta]$.

Moreover, β can be chosen so that, for each Borel set B in $V[G_\beta]$, if $A \cap B \cap V[G] \neq \emptyset$, then $A \cap B \cap V[G_\beta] \neq \emptyset$.

It suffices to see that (in either case), $A \subseteq V[G_\beta]$.

We will show that $A \cap V[G_{\beta+1}] \subseteq V[G_\beta]$. The argument for $A \cap V[G]$ is similar.

Names (I)

Cohen-names and random-names for elements of a Polish space X are given by Borel functions $f\colon 2^\omega \to X$; the realization of the name is $f(g)$, where g is the generic real.

Suppose that $1 \Vdash f(g) \notin V$. Then the f-preimage of each point is meager/null.

If $E \subseteq X$ is universally meager, then $f^{-1}[E]$ is meager.

If $E \subseteq X$ is universally null, then $f^{-1}[E]$ is Lebesgue null.

Names (II)

It follows that:

- If E is universally meager then for any nonmeager Borel B there is a nonmeager Borel $B' \subseteq B$ such that

$$f[B'] \cap E = \emptyset.$$

- If E is universally null then for any non-null Borel B there is a non-null Borel $B' \subseteq B$ such that

$$f[B'] \cap E = \emptyset.$$

Analytic sets and Borel sets

In either case:

$f[B']$ is analytic, so there exist Borel sets F_α $(\alpha < \omega_1)$ such that

$$f[B'] = \bigcup_{\alpha<\omega_1} F_\alpha$$

in any ω_1-preserving outer model.

B' then forces that $g \in B'$, so $f(g)$ will be in some F_α (all of which are disjoint from E).

Proof (IV)

We have then that (in either case) every element of X in

$$V[G_{\beta+1}] \setminus V[G_\beta]$$

is an element of a Borel set in $V[G_\beta]$ disjoint from

$$A \cap V[G_\beta],$$

and therefore also disjoint from A.

A version of the same argument works to show that $A \subseteq V[G_\beta]$.

Note that neither argument proceeded by taking sets which were universally null and not universally meager (or the reverse) and forcing to make the set not universally null (meager).

I don't know if it is possible to do this (ever or in general).

Universally measurable and categorical sets

We can rerun the same analysis with universally measurable and universally categorical sets.

Since universal measurability and categoricity of A are Π_2 properties over $(H(\aleph_1), \in, A)$:

If $A \subseteq X$ in $V[G]$ is universally categorical (or measurable), then there is a $\beta < \omega_2^V$ such that $A \cap V[G_\beta]$ is universally categorical (or measurable) in $V[G_\beta]$.

Moreover, β can be chosen so that, for each Borel set B in $V[G_\beta]$, if $A \cap B \cap V[G] \neq \emptyset$, then $A \cap B \cap V[G_\beta] \neq \emptyset$ (and similarly for $X \setminus A$).

Names (III)

Fix now a condition B and a Borel function $f\colon B \to X$ representing a name for an element of X.

If $E \subseteq X$ is universally categorical, then $f^{-1}[E]$ is has the Baire Property, so there is a nonmeager Borel $B' \subseteq B$ such that $f[B']$ is either contained in or disjoint from E.

If $E \subseteq X$ is universally measurable, then $f^{-1}[E]$ is Lebesgue measurable, so there is a non-null Borel $B' \subseteq B$ such that $f[B']$ is either contained in or disjoint from E.

In either case, B' forces that $f(g)$ will be in a ground model Borel set which is either contained in or disjoint from E.

Borel reinterpretations (I)

Definition. Given a subset A of a Polish space X, a partial order $\mathbb{P}$ and a V-generic filter G, the *Borel reinterpretation* of A in $V[G]$ is the union of all the (reinterpreted) ground model Borel sets contained in A.

Proposition. A subset A of a Polish space X is universally measurable if and only if, whenever $V[G]$ is an extension of V via random forcing, the Borel reinterpretations of A and $X \setminus A$ in $V[G]$ are complements.

Proposition. A subset A of a Polish space X is universally categorical if and only if, whenever $V[G]$ is an extension of V via Cohen forcing, the Borel reinterpretations of A and $X \setminus A$ in $V[G]$ are complements.

(Now prove Recław's Theorem.)

$\mathbb{G}_0$ graphs again

It follows that iterated random forcing (over a model of GCH) preserves the fact that the universally measurable coloring number of $\mathbb{G}_0$-graphs is at most 3 (and also that there are universally measurable sets without the Baire property).

We shall see that the same holds for Sacks forcing.

What are the universally measurable coloring numbers of $\mathbb{G}_0$-graphs in the Laver model?

(presumably there is a nonmeasurable universally categorical set in the Cohen model)

Cohen and random (III)

Theorem. (Larson-Neeman-Shelah) In an extension of a model of GCH by a length-ω_2 finite-support iteration of random forcing, there are only continuum many universally measurable sets, and a set A is universally measurable if and only if A and its complement are unions of $\aleph_1$-many Borel sets.

Theorem. In an extension of a model of GCH by a length-ω_2 finite-support iteration of Cohen forcing, there are only continuum many universally categorical sets, and a set A is universally measurable if and only if A and its complement are unions of $\aleph_1$-many Borel sets.

Borel reinterpretations (II)

Theorem. The Borel reinterpretation of a universally mesaurable set in an extension by (iterated) random forcing is universally measurable.

To see this (for a single random real), fix a universally measurable $A \subseteq X$ (for some Polish space X) a Borel

$$f : 2^\omega \to \mathrm{Meas}(X)$$

(representing a Borel measure in the forcing extension) and a Borel set $D \subseteq 2^\omega$ of positive measure (a condition in random forcing).

Define the Borel measure μ on 2^ω by

$$\mu(E) = \int f(x)(E)\, d\lambda,$$

where λ is Lebesgue measure on 2^ω.

Borel reinterpretations (III)

Since A is universally measurable, there exist Borel B and N such that $A \triangle B \subseteq N$ and $\mu(N) = 0$.

Then $\{x : f(x)(N) > 0\}$ has measure 0, and subtracting it from D we get a condition D' forcing that $f(g)(N) = 0$.

Letting $\hat{A}$ be the Borel reinterpretation of A in the forcing extension, it suffices to see that $\hat{A} \triangle B \subseteq N$ (where B and N are reinterpreted).

This in turn follows from the fact that if E is a Borel set contained in A, then $E \setminus B \subseteq A \setminus B \subseteq N$, and if E is disjoint from A, then $B \cap E \subseteq B \setminus A \subseteq N$.

References

P. Larson, I. Neeman, S. Shelah, *Universally measurable sets in generic extensions*, Fundamenta Mathematicae 208 (2010) 2, 173-192

B. Miller, *Measurable chromatic numbers*, J. Symbolic Logic 73 (2008), no. 4, 1139–1157