Nonmeasurability versus complete nonmeasurability

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$\mathbb{R}$ - real line,

$\mathbb{I}$ - a σ-ideal of subsets of $\mathbb{R}$

- containig singletons, i.e. $[\mathbb{R}]^\omega \subseteq \mathbb{I}$,
- with Borel base, i.e. $(\forall I \in \mathbb{I})(\exists B \in \mathrm{Borel} \cap \mathbb{I})(I \subseteq B)$,
- translation invariant, i.e. $(\forall I \in \mathbb{I})(\forall x \in \mathbb{R})(x + I = \{x + i \ : \ i \in I\} \in \mathbb{I})$.

Definition

Let $N \subseteq \mathbb{R}$. We say that the set N is

1. $\mathbb{I}$-*nonmeasurable* iff
$$N \notin \mathrm{Borel}[\mathbb{I}] = \{B \triangle I \ : \ B \in \mathrm{Borel}, I \in \mathbb{I}\};$$
2. *completely* $\mathbb{I}$-*nonmeasurable* iff
$$(\forall A \in \mathrm{Borel} \setminus \mathbb{I})(A \cap N \notin \mathbb{I} \ \wedge \ A \cap (\mathbb{R} \setminus N) \notin \mathbb{I}).$$

Remark

$N \subseteq \mathbb{R}$ is completely $\mathbb{I}$-nonmeasurable iff

$$(\forall A \in \mathrm{Borel} \setminus \mathbb{I})(A \cap N \neq \emptyset \ \wedge \ A \cap (\mathbb{R} \setminus N) \neq \emptyset).$$

Remark

- N is completely $\mathbb{L}$-nonmeasurable if $\lambda_*(N) = 0$ and $\lambda_*(\mathbb{R} \setminus N) = 0$.
- The definition of completely $\mathbb{K}$-nonmeasurability is equivalent to the definition of completely Baire-nonmeasurability.
- N is completely $[\mathbb{R}]^\omega$-nonmeasurable iff N is a Bernstein set.

Question

Let $\mathcal{P} \subseteq \mathbb{I}$ be a partition of $\mathbb{R}$. Is it possible that

$$(\forall \mathcal{A} \subseteq \mathcal{P})(\bigcup \mathcal{A} \text{ is } \mathbb{I}\text{-nonmeasurable} \rightarrow$$

$$\bigcup \mathcal{A} \text{ is completely } \mathbb{I}\text{-nonmeasurable})?$$

Definition

$\mathbb{I}$ has *Steinhaus property* if

$$(\forall A \in \mathrm{Borel} \setminus \mathbb{I})(\forall B \notin \mathbb{I})(A - B \text{ contains an open interval})$$

where

$$A - B = \{a - b \ : \ a \in A, \ b \in B\}.$$

Theorem

Assume $\mathbb{I}$ has Steinhaus property. Then there exists a partition $\mathcal{P} \subseteq \mathbb{I}$ of $\mathbb{R}$ such that for every $\mathcal{A} \subseteq \mathcal{P}$

$$\bigcup \mathcal{A} \text{ is } \mathbb{I}\text{-nonmeasurable} \to \bigcup \mathcal{A} \text{ is completely } \mathbb{I}\text{-nonmeasurable.}$$

Proof.

For $x, y \in \mathbb{R}$ let $x \approx y \leftrightarrow x - y \in \mathbb{Q}$. Let $\mathcal{P} = \mathbb{R}/\approx = \{x_\alpha + \mathbb{Q} : \alpha \in 2^\omega\}$.

Take $\mathcal{A} \subseteq \mathcal{P}$ such that $|\mathcal{A}| > \omega$ and $|\mathcal{P} \setminus \mathcal{A}| > \omega$.

Assume that $\mathcal{A}$ is not completely $\mathbb{I}$-nonmeasurable.

Then $\bigcup \mathcal{A} \notin \mathbb{I}$ and $\bigcup(\mathcal{P} \setminus \mathcal{A}) \notin \mathbb{I}$ and at least one of this sets contains $\mathbb{I}$-possitive Borel set.

So $\bigcup \mathcal{A} - \bigcup(\mathcal{P} \setminus \mathcal{A})$ contains an open interval.

$\bigcup \mathcal{A} - \bigcup(\mathcal{P} \setminus \mathcal{A}) = \{x_\alpha - x_\beta + \mathbb{Q} : x_\alpha + \mathbb{Q} \in \mathcal{A},\ x_\beta + \mathbb{Q} \in \mathcal{P} \setminus \mathcal{A}\}$.

$\bigcup \mathcal{A} - \bigcup(\mathcal{P} \setminus \mathcal{A}) \cap \mathbb{Q} = \emptyset$. Contradiction. □

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1. *there is $\mathcal{A} \subseteq \mathcal{P}$ such that $\bigcup \mathcal{A}$ is completely $\mathbb{I}$-nonmeasurable;*
2. *there is $\mathcal{A} \subseteq \mathcal{P}$ such that $\bigcup \mathcal{A}$ is $\mathbb{I}$-nonmeasurable but is not completely $\mathbb{I}$-nonmeasurable.*

Proof.

1. Standard construction of Bernstein set.
2. Let $\mathcal{P} = \{Y_\alpha : \alpha \in 2^\omega\}$, $Y_\alpha = \{y_0^\alpha, y_1^\alpha, \ldots, y_n^\alpha\}$, $y_0^\alpha < y_1^\alpha < \ldots < y_n^\alpha$.
   Define $X_k = \{y_k^\alpha : \alpha \in 2^\omega\}$. Wlog $X_0 \notin \mathbb{I}$.
   We can find $r \in \mathbb{R}$ such that $X_0 \cap (-\infty, r) \notin \mathbb{I}$ and $X_0 \cap (r, +\infty) \notin \mathbb{I}$.
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Assume that $\mathcal{P} \subseteq [\mathbb{R}]^\omega$ *is a partition of* $\mathbb{R}$*. Then we can find* $\mathcal{A} \subseteq \mathcal{P}$ *such that* $\bigcup \mathcal{A}$ *is completely* $[\mathbb{R}]^\omega$*-nonmeasurable.*

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Theorem ($\neg CH$)

Assume that $\mathcal{P} \subseteq [\mathbb{R}]^\omega$ is a partition of $\mathbb{R}$. Then we can find $\mathcal{A} \subseteq \mathcal{P}$ such that $\bigcup \mathcal{A}$ is $[\mathbb{R}]^\omega$-nonmeasurable but is not completely $[\mathbb{R}]^\omega$-nonmeasurable.

Proof.

Take $\mathcal{A} \subseteq \mathcal{P}$ such that $|\mathcal{A}| = \omega_1$.
$|\bigcup \mathcal{A}| = \omega_1 < 2^\omega$. So, $\bigcup \mathcal{A}$ is $[\mathbb{R}]^\omega$-nonmeasurable.
Fix $\{Q_\alpha \ : \ \alpha \in 2^\omega\}$ a family of pairwise disjoint perfect sets.
There exists α such that $Q_\alpha \cap \bigcup \mathcal{A} = \emptyset$. So, $\bigcup \mathcal{A}$ is not completely $[\mathbb{R}]^\omega$-nonmeasurable. □

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There is $\mathcal{P} \subseteq [\mathbb{R}]^\omega$ a partition of $\mathbb{R}$ such that for any $\mathcal{A} \subseteq \mathcal{P}$

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Corollary

TFAE:

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2. *there is* $\mathcal{P} \subseteq [\mathbb{R}]^\omega$ *a partition of* $\mathbb{R}$ *such that for any* $\mathcal{A} \subseteq \mathcal{P}$

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$$\Updownarrow$$

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Thank You for Your Attention

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