On algebraic sums, trees and null sets in the Cantor space and the Baire space

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We will work in the Cantor space 2^ω and the Baire space $\mathbb{Z}^\omega$, both equipped with coordinate-wise addition $+$.

For $A, B \subseteq 2^\omega$(or $\mathbb{Z}^\omega$) the algebraic sum of A and B is the set

$$A + B = \{a + b : a \in A \wedge b \in B\}.$$

Let $T \subseteq B^{<\omega}$, $B \in \{2, \mathbb{Z}\}$ be a tree. We will use the following notions related to trees:

- $\mathrm{succ}_T(\sigma) = \{b \in B : \sigma^\frown b \in T\}$;
- $\mathrm{split}(T) = \{\sigma \in T : |\mathrm{succ}_T(\sigma)| \geq 2\}$;
- $\omega\text{-}\mathrm{split}(T) = \{\sigma \in T : |\mathrm{succ}_T(\sigma)| = \omega\}$.

Definition

A tree $T \subseteq B^{<\omega}$, $B \in \{2, \mathbb{Z}\}$, is called

- a Sacks or perfect tree, if $(\forall \sigma \in T)(\exists \tau \in T)(\sigma \subseteq \tau \wedge \tau \in \mathrm{split}(T))$;
- a uniformly perfect tree, if it is perfect and $(\forall n \in \omega)(B^n \cap T \subseteq \mathrm{split}(T) \vee B^n \cap T = \emptyset)$;
- a ($\omega-$)Silver tree, if there are $A \subseteq [\omega]^\omega$ and $x \in 2^\omega$ ($\mathbb{Z}^\omega$) such that $T = \{\sigma \in 2^{<\omega}(\mathbb{Z}^{<\omega}) : (\forall n \in \mathrm{dom}(\sigma) \cap \omega \setminus A)(\sigma(n) = x(n))\}$;
- a splitting tree, if $(\forall \sigma \in T)(\exists N \in \omega)(\forall n > N)(\forall i \in 2)(\exists \tau \in T)(\sigma \subseteq \tau \wedge \tau(n) = i)$;
- a Miller tree, if $(\forall \sigma \in T)(\exists \tau \in T)(\sigma \subseteq \tau \wedge \tau \in \omega\text{-split}(T))$.

For a tree $T \subseteq B^{<\omega}$ the set

$$[T] = \{x \in 2^\omega : (\forall n)(x \restriction n \in T)\}.$$

denotes all infinite branches of T.

Definition

Let $\mathbb{T}$ be a family of some kind of trees. A tree ideal t_0 associated with $\mathbb{T}$ is defined as follows

$$A \in t_0 \iff (\forall T \in \mathbb{T})(\exists T' \subseteq T, T' \in \mathbb{T})([T'] \cap A = \emptyset).$$

Examples: Marczewski $(\sigma-)$ideal s_0 associated with Sacks trees, u_0 with uniformly perfect trees, v_0 with Silver trees.

Let $\mathcal{I}$ be a $\sigma-$ideal.

Definition

We call A an $\mathcal{I}-$Luzin set, if $|A| > |A \cap I|$ for each $I \in \mathcal{I}$.

For the $\sigma-$ideal $\mathcal{M}$ of meager sets $\mathcal{M}-$Luzin sets are known as generalized Luzin sets and for the $\sigma-$ideal $\mathcal{N}$ of null sets $\mathcal{N}-$Luzin sets are known as generalized Sierpiński sets.

Theorem (M., Żeberski 2015)

Let $\mathfrak{c}$ be a regular cardinal. Then for every generalized Luzin set $L \subseteq \mathbb{R}$ and every generalized Sierpiński set $S \subseteq \mathbb{R}$ it is the case that $L + S \in s_0$.

The essential part of the proof were lemmas of the following form.

Lemma

Let $\mathcal{I} \in \{\mathcal{M}, \mathcal{N}\}$. For $A \in \mathcal{I}$ and a perfect set P there exists a perfect set $P' \subseteq P$ such that

$$A + P' \in \mathcal{I}.$$

Let $\mathbb{T}$ be a family of some kind of trees and let $\mathcal{I}$ be a $\sigma-$ideal.

Goal

For every set $A \in \mathcal{I}$ and a tree $T \in \mathbb{T}$ there exists $T' \subseteq T$, $T' \in \mathbb{T}$ such that

$$A + \underbrace{[T'] + [T'] + \cdots + [T']}_{n-times} \in \mathcal{I}$$

for each $n \in \omega$.

During this talk we focus on the **measure case** and its neighborhood.

Definition

We call $A \subseteq 2^\omega$ a small set if there is a partition $\{I_n : n \in \omega\}$ of ω into intervals and a collection $(J_n : n \in \omega)$ such that $J_n \subseteq 2^{I_n}$ for $n \in \omega$ with $\sum_{n\in\omega} \frac{|J_n|}{2^{|I_n|}} < \infty$ and

$$A \subseteq \{x \in 2^\omega : (\exists^\infty n \in \omega)(x \restriction I_n \in J_n)\}.$$

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Theorem (Bartoszyński, 1988)

Every null subset of 2^ω is a union of two small sets.

Lemma

For every small set $F \subseteq 2^\omega$ and every Silver tree T there is a Silver tree $T' \subseteq T$ such that for every $n \in \omega$

$$F + \underbrace{[T'] + [T'] + \ldots + [T']}_{n-times} \text{ is small.}$$

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Proof.

- Let F be small with an associated partition $\{I_n : n \in \omega\}$ and a collection of legal sequences $\{J_n : n \in \omega\}$.

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- Let F be small with an associated partition $\{I_n : n \in \omega\}$ and a collection of legal sequences $\{J_n : n \in \omega\}$.
- Let T be a Silver tree with associated $A \in [\omega]^\omega$ and a pattern $x_T \in 2^\omega$.

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- Let F be small with an associated partition $\{I_n : n \in \omega\}$ and a collection of legal sequences $\{J_n : n \in \omega\}$.
- Let T be a Silver tree with associated $A \in [\omega]^\omega$ and a pattern $x_T \in 2^\omega$.
- Set $A' \subseteq A$ infinite such that $|A' \cap I_n| \leq 1$ for any $n \in \omega$.

  $\vdots$

Proof cntd.

⋮

- Set new sequences of legal patterns $J'_n = (J_n + x_T \restriction I_n) \cup J_n$ and $J''_n = J'_n + (00\dots010\dots))$ (1 on position $i \in A \cap I_n$).

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- Set new sequences of legal patterns $J'_n = (J_n + x_T \restriction I_n) \cup J_n$ and $J''_n = J'_n + (00\dots010\dots))$ (1 on position $i \in A \cap I_n$).
- Then $|J''_n| \leq 4|J_n|$, hence the set

$$\{x \in 2^\omega : (\exists^\infty n \in \omega)(x \restriction I_n \in J''_n)\}$$

is small. □

Theorem

For every null sets $F \subseteq 2^\omega$ and every Silver tree $T \subseteq 2^{<\omega}$ there exists a Silver tree $T' \subseteq T$ such that

$$F + \underbrace{[T'] + [T'] + \ldots + [T']}_{n-times} \in \mathcal{N}.$$

Lemma

Let $\sum_{n\in\omega} a_n = s < \infty$, where $a_n > 0$ for all $n \in \omega$. Then there is a nondecreasing sequence of natural numbers $(k_n)_{n\in\omega}$ such that for each $b \in \omega$

$$\sum_{n\in\omega} (2^b)^{k_n} a_n < 2^{b^2} s.$$

Theorem

For every small set $A \in 2^\omega$ and every (uniformly) perfect tree T there exists (uniformly) perfect tree $T' \subseteq T$ such that

$$A + \underbrace{[T'] + [T'] + \ldots + [T']}_{n\text{–times}} \text{ is small.}$$

Corollary

For every null set $A \in 2^\omega$ and every (uniformly) perfect tree T there exists (uniformly) perfect tree $T' \subseteq T$ such that

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Theorem

For every small (null) set $F \subseteq 2^\omega$ there is a splitting tree T such that for every $n \in \omega$ the set

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is small (belongs to $\mathcal{N}$).

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Question

Let F be a null subset of 2^ω. Is it true that for every splitting tree T there exists a splitting tree $T' \subseteq T$ such that

$$F + \underbrace{[T'] + [T'] + \ldots + [T']}_{n-times} \in \mathcal{N}?$$

Theorem

Let $F \in \mathcal{M} \cap \mathcal{N}$ and let T be a perfect (resp. uniformly perfect or Silver) tree. Then there is a perfect (resp. uniformly perfect or Silver) tree $T' \subseteq T$ such that

$$F + \underbrace{[T'] + [T'] + \ldots + [T']}_{n-times} \in \mathcal{M} \cap \mathcal{N}.$$

Lemma (Bartoszyński and Shelah, 1992)

For every set $E \in \mathcal{E}$ there is a partition $\{I_n : n \in \omega\}$ of ω to intervals and sets of finite sequences $J_n \subseteq 2^{I_n}$, $n \in \omega$, such that $\sum_{n\in\omega} \frac{1}{2^{|I_n|}} < \infty$, for each $n \in \omega$ $\frac{|J_n|}{2^{I_n}} \leq \frac{1}{2^n}$, and

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Theorem

For every set $E \in \mathcal{E}$ and every Silver (resp. perfect or uniformly perfect) tree T there is a Silver (resp. perfect or uniformly perfect) tree $T' \subseteq T$ such that for every $n \in \omega$

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Definition

We will say that a set A is fake null, denote by $A \in f\mathcal{N}$, if

$$(\forall \varepsilon > 0)(\exists (\sigma_n : n \in \omega)) \left(\sum_{n \in \omega} \frac{1}{2^{|\sigma_n|}} < \varepsilon \,\&\, A \subseteq \bigcup_{n \in \omega} [\sigma_n] \right).$$

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Couple of properties of $f\mathcal{N}$

- It is a translation invariant $\sigma-$ideal.
- It is orthogonal to $\mathcal{M}$, i.e. there exists comeager $G \in f\mathcal{N}$.
- There is a compact set which is not fake null, e.g. any full binary tree.

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- There is a compact set which is not fake null, e.g. any full binary tree.
- More at the Łukasz's talk!

Lemma (Essentially Bartoszyński and Judah, 1995)

Let $F \in f\mathcal{N}$. *Then there is a sequence* $(S_n : n \in \omega)$, $S_n \subseteq \mathbb{Z}^n$ *for each* $n \in \omega$, *such that* $\sum_{n\in\omega} \frac{|S_n|}{2^n} < \infty$ *and*

$$F \subseteq \{x \in \mathbb{Z}^\omega : (\exists^\infty n \in \omega)(x \restriction n \in S_n)\}.$$

Conversely, if $(S_n : n \in \omega)$, $S_n \subseteq \mathbb{Z}^n$, *satisfy* $\sum_{n\in\omega} \frac{|S_n|}{2^n} < \infty$, *then*

$$\{x \in \mathbb{Z}^\omega : (\exists^\infty n \in \omega)(x \restriction n \in S_n)\} \in f\mathcal{N}.$$

Theorem

For every $F \in f\mathcal{N}$ and every (uniformly) perfect tree $T \subseteq \mathbb{Z}^{<\omega}$ there is a (uniformly) perfect tree $T' \subseteq T$ such that for each n

$$F + \underbrace{[T'] + [T'] + \cdots + [T']}_{n-times} \in f\mathcal{N}.$$

Theorem

$[T_1] + [T_2] \notin f\mathcal{N}$ *for any Miller trees* T_1, T_2.

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Every Miller tree T *contains a Miller tree* T' *such that* $[T'] \in f\mathcal{N}$.

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Proposition

Every Miller tree T *contains a Miller tree* T' *such that* $[T'] \in f\mathcal{N}$.

Corollary

There exists a fake null set F *such that for any Miller tree* T
$F + [T] \notin f\mathcal{N}$.

Corollary

There exists a fake null set F *such that for any* $\omega-$*Silver tree* T
$F + [T] \notin f\mathcal{N}$.

Thank you!

- Michalski Marcin, Rałowski Robert, Żeberski Szymon, *On algebraic sums, trees and ideals in the Cantor space*, arXiv:2405.13775.
- Mazurkiewicz Łukasz, Michalski Marcin, Rałowski Robert, Żeberski Szymon, *On algebraic sums, trees and ideals in the Baire space*, arXiv:2409.17748.