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WS 2025

GENERALIZED WAŻEWSKI DENDRITES, GENERIC SUBCONTINUA, AND GENERIC CHAINS

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Introduction

Definition 1.1

A compact, connected, and metrizable space is a **continuum**. If it is locally connected, it is a **Peano continuum**.

Definition 1.2

A continuum X is **hereditarily equivalent** if every non-degenerate subcontinuum of it is homeomorphic to X. In this case we say X is HEC.

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If X is a continuum, $\mathrm{Cont}(X)$ is the hyperspace of subcontinua of X. $\mathrm{Fin}_1(X)$ is the hyperspace of singletons of X.

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so $\mathscr{G}$ is a comeager subset of $\mathrm{Cont}(X)$.

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Generically Hereditarily Equivalent Continuum

Definition 1.5

A continuum X is **generically hereditarily equivalent** if

$$\{K \in \mathrm{Cont}(X) \mid K \simeq X\}$$

is comeager in $\mathrm{Cont}(X)$. In this case we say X is GHEC.

Generalized Ważewski dendrites are GHEC

Universal Ważewski dendrite

Definition 2.1

A **dendrite** is a Peano continuum without homeomorphic copies of the circle.

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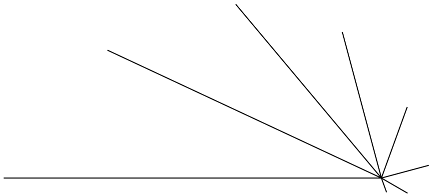
The universal Ważewski dendrite is such that every dendrite can be embedded into it. How can one construct it?



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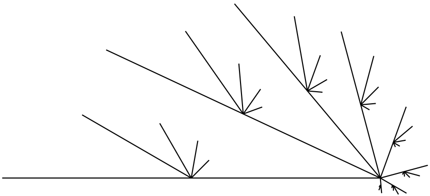

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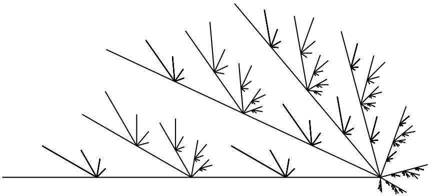

Order

Definition 2.2

Given a topological space X and $A \subseteq X$, the **order** of A in X is the least cardinal number α for which every open set $U \supseteq A$ there exists an open set V such that

$$A \subseteq V \subseteq U \text{ and } |\partial V| \leq \alpha.$$

We write that

$$\operatorname{ord}(A, X) = \alpha.$$

Definition 2.3

Let $M \subseteq \{3, 4, 5, \dots\} \cup \{\omega\}$. The dendrite W_M is defined as the dendrite whose set of branching points are of order $m \in M$ and for all $m \in M$

$$\{x \in W_M \mid \operatorname{ord}(x, W_M) = m\}$$

is arcwise dense in W_M.

Maximal branching points

Definition 2.4

Let X, Y be dendrites with $X \subseteq Y$. The first point map $r_{Y,X} : Y \to X$ takes $y \in Y$ to the first point in the arc starting from y to any $x \in X$ that is also in X.

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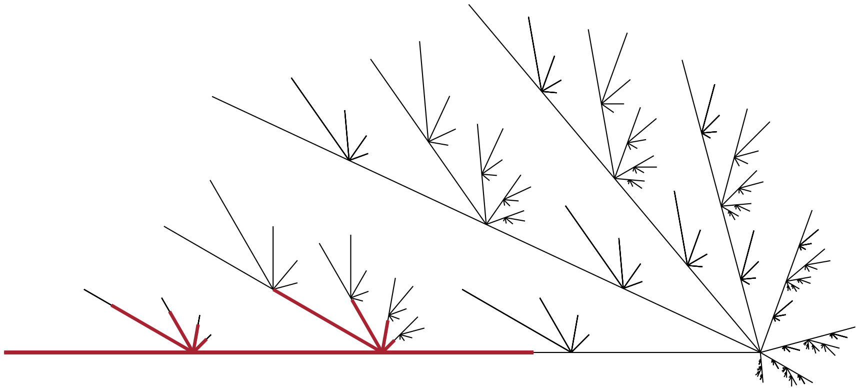

Maximal branching points

Lemma 2.5

Let X, Y be dendrites with $X \subseteq Y$. A branching point x of X is **maximal** in Y if one of the following are satisfied:

(i) $|r_{Y,X}^{-1}(x)| = 1$

(ii) there is no arc $A \subseteq Y$ from $y \in Y \setminus X$ to x with $A \cap X = \{x\}$.

(iii) every open neighborhood of x in X meets each component of $Y \setminus \{x\}$.

Full subdendrite

Definition 2.6

Given a dendrite X, $\mathcal{B}(X)$ is the collection of branching points of X.

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Definition 2.7

Let Y be a dendrite. If $X \subseteq Y$ is a non-degenerate subdendrite, we say that X is full if for every $b \in \mathcal{B}(Y) \cap X$, it holds that $b \in \mathcal{B}(X)$ and b is maximal in Y. We denote

$$\mathsf{Full}(Y) = \{K \in \mathrm{Cont}(Y) \mid K \text{ is full in } Y\}.$$

Generalized Ważewski dendrites are GHEC

Proposition 2.8

If X is a dendrite, then $\mathsf{Full}(X)$ is a G_δ dense subset of $\mathrm{Cont}(X)$.

Theorem 2.9

For every $K \in \mathsf{Full}(W_M)$ it holds that $K \simeq W_M$ and

$$\mathsf{End}(K) \cap \mathcal{B}(W_M) = \emptyset.$$

Corollary 2.10

W_M is GHEC.

Additional restriction

Proposition 2.11

The collection of nowhere dense subcontinua of W_M is a G_δ dense subset of $\mathrm{Cont}(W_M)$.

The almost complete characterization

Theorem 2.12

If X is a Peano GHEC, then X is an arc or a dendrite such that for every $m \in \{3, 4, \dots\} \cup \{\omega\}$ the collection of branching points of order m is either dense or empty. Moreover, the collection of branching points is arcwise dense.

Strongly GHEC

Definition 2.13

A continuum X is strongly GHEC if

$$\{K \in \mathrm{Cont}(X) \mid K \simeq X\}$$

contains a comeager subset $\mathscr{H}$ of $\mathrm{Cont}(X)$ such that whenever $K_1, K_2 \in \mathscr{H}$ there exists a homeomorphism $\psi : X \to X$ such that $\psi(K_1) = K_2$.

Theorem 2.14

W_M is strongly GHEC. Precisely, the collection of subcontinua K of W_M satisfying

(i) $K \in \mathrm{Full}(W_M)$.

(ii) K is nowhere dense in W_M.

is G_δ dense and for any pair K_1, K_2 in this collection there exists a homeomorphism $\psi : W_M \to W_M$ with $\psi(K_1) = K_2$.

Generic chains

Definition 3.1

An **order arc** in $\mathrm{Cont}(X)$ is a subcontinuum $\mathcal{A} \subseteq \mathrm{Cont}(X)$ homeomorphic to $[0,1]$ such that for every pair of points $A, B \in \mathcal{A}$, either $A \subseteq B$ or $B \subseteq A$. We denote the space of order arcs by

$$\mathsf{OA}(X) \subseteq \mathrm{Cont}(\mathrm{Cont}(X)).$$

Definition 3.2

A **maximal order arc** in $\mathrm{Cont}(X)$ is an order arc starting from a set $\{x\}$ for some $x \in X$ and ending in X. We can restrict $\mathsf{OA}(X)$ space to the collection of maximal order arcs, denoted by $\mathsf{MOA}(X)$, which is a Polish space.

Definition 3.3

Let X be a continuum, we say that

- GCHEC holds for X if and only if

$$\{\mathcal{C} \in \mathrm{MOA}(X) \mid \forall C \in \mathcal{C}, C \text{ nondegenerate implies } C \simeq X\}$$

  is a comeager subset of $\mathrm{MOA}(X)$.

- GCGHEC holds for X if and only if

$$\{\mathcal{C} \in \mathrm{MOA}(X) \mid \forall^* C \in \mathcal{C}\ (C \simeq X)\}$$

  is a comeager subset of $\mathrm{MOA}(X)$.

Disintegration Theorem

Theorem 3.4

If Z and Y are Polish spaces and $f : Z \to Y$ in a continuous and comeager way, then a set S with Baire property is comeager in Z if and only if $S \cap f^{-1}(y)$ is comeager in $f^{-1}(y)$ for comeager many y in Y.

Melleray, J., Tsankov, T. Generic representations of abelian groups and extreme amenability. Isr. J. Math. 198, 129–167 (2013).
https://doi.org/10.1007/s11856-013-0036-5

GCHEC holds for W_M

Lemma 3.5

If $\mathcal{C} \in \mathrm{MOA}(W_M)$ and the collection of $K \in \mathcal{C}$ with $K \simeq W_M$ is dense in $\mathcal{C}$, then every nondegenerate element of $\mathcal{C}$ is homeomorphic to W_M.

Theorem 3.6

GCHEC holds for W_M.

Properties of the comeager maximal order arc in $\mathrm{MOA}(W_M)$

Willful chain

Definition 4.1

A chain $\mathcal{C} \in \text{MOA}(W_M)$ is **willful** if for every arc $A \subseteq W_M$ and $K_1, K_2 \in \mathcal{C}$ with

(i) $K_1 \subsetneq K_2$,

(ii) $\emptyset \neq K_1 \cap A \subsetneq A$,

then $K_1 \cap A \subsetneq K_2 \cap A$.

Theorem 4.2

The set of chains $\mathcal{C} \in \mathrm{MOA}(W_M)$ satisfying

(i) The root of $\mathcal{C}$ is an endpoint of W_M.

(ii) If $K, L \in \mathcal{C}$ with $K \subsetneq L$, then K is nowhere dense in L.

(iii) If $K \in \mathcal{C}$, then $|\,\mathrm{End}(K) \cap \mathcal{B}(W_M)| \leq 1$.

(iv) $\mathcal{C}$ is willful.

form a comeager set. Moreover, any two chains satisfying these conditions are ambiently equivalent.

Generic chain of MOA(W_M)

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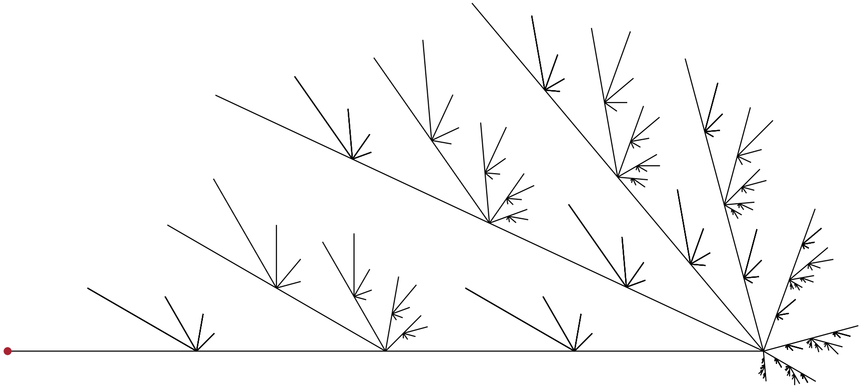
$\mathcal{C}_1, \mathcal{C}_2 \in \mathrm{MOA}(X)$ are ambiently equivalent if there exists $h \in \mathrm{Homeo(X)}$ for which

$$\{h(K) \mid K \in \mathcal{C}_1\} = \mathcal{C}_2$$

The idea



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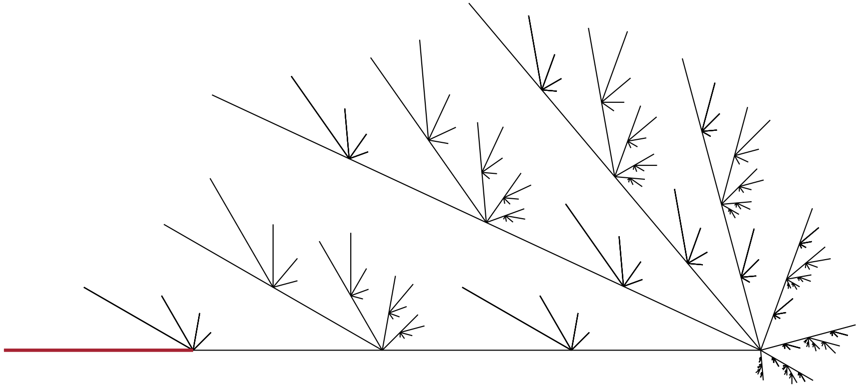

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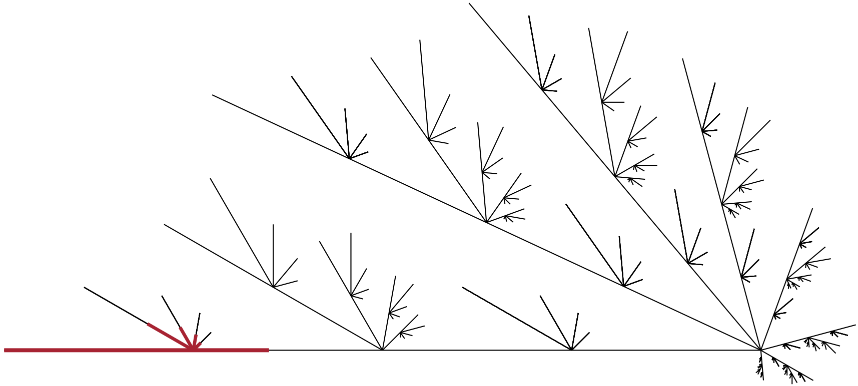

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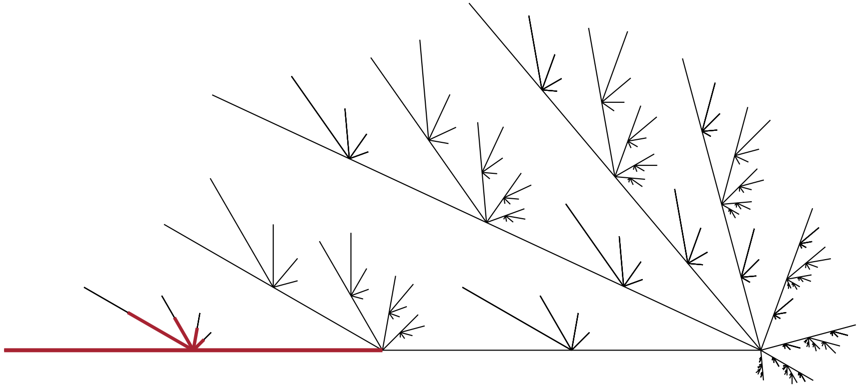

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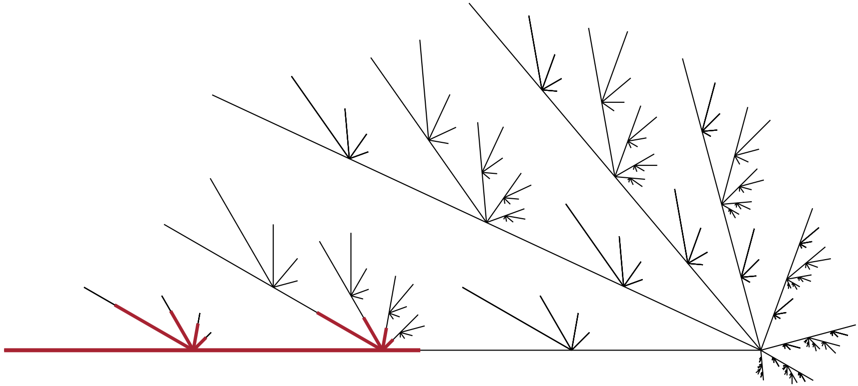

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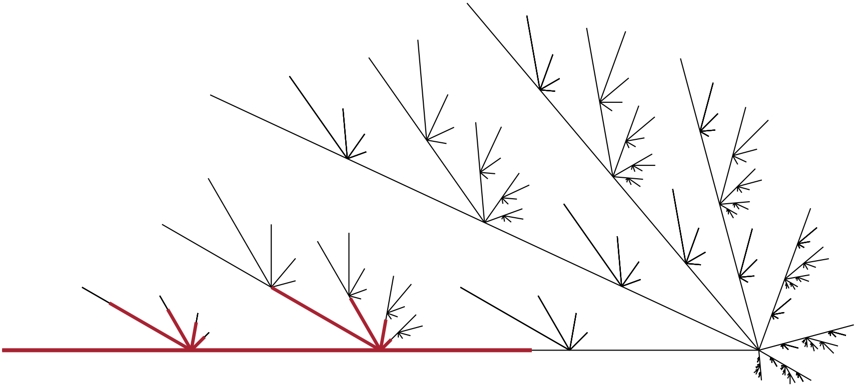

Děkuju!