Peripherally Hausdorff spaces and fixed-point theorem

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Theorem (Banach fixed-point theorem, 1920)

Every Lipschitz contraction on complete metric space has unique fixed point.

Here $f : X \to X$ is a Lipschitz contraction iff existst $c \in [0, 1)$ s.t. for every $x, y \in X$

$$d(f(x), f(y)) \leq c \cdot d(x, y).$$

Feebly topological contraction

Definition (Kupka)

Let X - T_0 topological space, then $f : X \to X$ is feebly topological contraction if for each open cover $\mathcal{U}$ we have

$$\forall x, y \in X \, \exists n \in \omega \, \exists U \in \mathcal{U} \;\; f^n[\{x, y\}] \subseteq U$$

Theorem (Kupka, 1998)

If X top. space $f : X \to X$ s.t.

- *f has closed graph,*
- *f is feebly top. contraction*

then f has fixed point. Moreover, if X is T_1 then fixed point is unique.

Corollary

If X is a Hausdorff topological space and f is a continuous feebly topological contraction on X, then f has a unique fixed point.

Theorem

If X is a Hausdorff first-countable topological space and f is a closed feebly topological contraction on X, then f has a unique fixed point.

Remark

First countability can not be dropped.

For $r \in \mathbb{Z}$ and $A \subseteq \mathbb{N}$ ($\mathbb{N}$ strictly positive integers), let

$$A + r := \{a + r : a \in A\} \cap \mathbb{N}.$$

Let $r \in \mathbb{Z}$ and let $\mathcal{G}$ be a family of subsets of $\mathbb{N}$. Let

$$\mathcal{G} + r := \{G + r : G \in \mathcal{G}\},$$

$$S(\mathcal{G}) := \{\mathcal{G} - n : n \in \omega\}$$

Define $\{\mathcal{F}_\alpha : \alpha < \mathfrak{c}\}$ ultrafilters and $\{C_\alpha : \alpha < \mathfrak{c}\}$ infinite subsets of $\mathbb{N}$ s.t.

- if $\alpha \neq \beta$ then $\mathcal{S}(\mathcal{F}_\alpha) \cap \mathcal{S}(\mathcal{F}_\beta) = \emptyset$,
- $C_\alpha \in \mathcal{F}_\alpha$ ans $C_\alpha \cap (C_\alpha - m)$ is finite for every $\alpha < \mathfrak{c}$ and $m \in \mathbb{N}$ (positive integer),
- each infinite subset of $\mathbb{N}$ is element of some $\mathcal{F}_\alpha$.

$$X = \mathbb{N} \cup \{0\} \cup \bigcup_{\alpha < \mathfrak{c}} \mathcal{S}(\mathcal{F}_\alpha)$$

Define $f : X \to X$ s.t.

$$f(x) = \begin{cases} n+1 & n \in \{0\} \cup \mathbb{N}, \\ 0 & x = \mathcal{F}_\alpha,\ \alpha < \mathfrak{c}, \\ \mathcal{F}_\alpha - (n-1) & x = \mathcal{F}_\alpha - n,\ n \in \mathbb{N} \wedge \alpha < \mathfrak{c}. \end{cases}$$

f is closed map and feebly topological contraction (but not continuous) without fixed point.

Lacally Hausdorff space

Definition

A topological space X is *locally Hausdorff* if every point of the space has an open neighbourhood U such that the topology of X restricted to U is Hausdorff.

Theorem

If X is a locally Hausdorff T_1 topological space and f is a continuous feebly topological contraction on X, then f has a unique fixed point.

Example

$X = \mathbb{Z}$, base:
if $n \in \omega$ then $U = \{-n\} \cup A$ where $A \subseteq \mathbb{N}$ is cofinite in $\mathbb{N}$,
if $n > 0$ $U = \{n\}$.
X is locally Hausdorff space.

$$\forall n \in \mathbb{Z}\ f(n) = n + 1.$$

f is feebly topological contraction and closed map without fixed point.

Peripherally Hausdorff space

Definition

For every $\alpha \in On$ define a class $\mathcal{F}_\alpha$ as follows: for every T_1 topological space X, we say that $X \in \mathcal{F}_\alpha$ is α-Hausdorff space if

if $\alpha = 0$ then $X = \{x\}$ and,

if $\alpha > 0$ then $\forall x \in X \exists \beta < \alpha \; [x] \in \mathcal{F}_\beta$ where

$$[x] = \bigcap \{cl(U) : \; x \in U - \text{ is open in } X\}.$$

We say that X is peripherally Hausdorff iff $\exists \alpha \in On \, X \in \mathcal{F}_\alpha$,
We have

- If $\beta \leq \alpha$ then $\mathcal{F}_\beta \subseteq \mathcal{F}_\alpha$,
- $X \in \mathcal{F}_1$ iff X is a Hausdorff space.

Example (Niemytzki like half-plane)

Set $X = \mathbb{R} \times [0, \infty)$. Topological subbase of X is defined as follows, for points with positive second axis we equipe euclidean neighbourhoods. For points of form $(x, 0)$ where $x \in \mathbb{R}$ and $\alpha \in (0, \pi/2)$ we define

$$U_\alpha(x,0) = \{(r,s) \in X : \ (r,s) = (x,0) \vee (s \neq |x| \wedge \operatorname{tg}\alpha \cdot |r| < s)\}.$$

It is easy to see that

$$[(x,y)] = \begin{cases} \{(x,y)\} \cup \{(x,0)\} & \text{for } y > 0 \\ \mathbb{R} \times \{0\} \cup \{x\} \times (0,\infty) & \text{for } y = 0 \end{cases},$$

which is T_2 space.
Then X is 2-Hausdorff, locally Hausdorff space but not T_2.

Definition (Hausdorff rank)

Let X-peripherally Hausdorff space, define Hausdorff rank of X

$$rank_H(X) = \min\{\alpha \in On : \ X \in \mathcal{F}_\alpha\}.$$

Theorem

For every $\alpha \in On$ there is peripherally Hausdorff space X s.t. $\alpha \leq rank_H(X)$.

Theorem

If X, Y are peripherally Hausdorff spaces then

$$rank_H(X \times Y) = \max\{rank_H(X), rank_H(Y)\}.$$

Theorem

There are peripherally Hausdorff spaces $\{X_n : n \in \omega\}$ s.t for each $n \in \omega$ $rank(X_n) = n$ and $\prod_{n\in\omega} X_n$ is not peripherally Hausdorff space.

Feebly$^+$ topological contraction

Definition

Let X - topological space, then $f : X \to X$ is feebly$^+$ topological contraction if for each open cover $\mathcal{U}$ we have

$$\forall x, y \in X \, \exists U \in \mathcal{U} \, \forall^\infty n \in \omega \;\; f^n[\{x, y\}] \subseteq U$$

Theorem

For every peripherally Hausdorff space X, every continuous weak$^+$ topological contraction on X has unique fixed-point.

Example

$$X = \{-1\} \cup [0, 1].$$

Let the base of X consist of all sets of the form:

- $J \cap [0, 1]$, where J is an open interval, and
- $((L \setminus \{0\}) \cap X) \cup \{-1\}$, where L is an open interval containing 0.

Let $f : X \to X$ be defined by

$$f(x) = \frac{1}{2} \cdot x \;\text{ where } x \in [0, 1] \;\text{ and } f(-1) = 0.$$

Then X is a compact peripherally Hausdorff (in fact 2-Hausdorff) space and f is a continuous weak$^+$ contraction but $f \subseteq X \times X$ is not closed. Of course, the point 0 is a fixed point of f.

Weak* topologies

Let X be a linear topological space, $\mathsf{E} = \{\mathsf{s}_1, \dots, \mathsf{s}_n\}$ be a finite set of seminorms on X, $x \in X$ and $\varepsilon > 0$. Let

$$V(x; \mathsf{E}, \varepsilon) := \{z \in X : \mathsf{s}_1(x - z) < \varepsilon, \dots, \mathsf{s}_n(x - z) < \varepsilon\}.$$

Theorem (Lebesgue's analogue lemma)

Let

1. X *linear space over reals,*
2. S *is a family of seminorms separating points in* X*,*
3. $Y \subseteq X$ *is compact in the weak topology* τ *determined by* S*,*
4. $\mathcal{U}$ *is a* τ*-open cover of* Y*,*

then then there exist a finite set $\mathsf{E} \subseteq \mathsf{S}$ *and* $\varepsilon > 0$ *such that for each* $x \in Y$ *there exists* $U \in \mathcal{U}$ *such that* $V(x; \mathsf{E}, \varepsilon) \subseteq U$*.*

Theorem

If

1. X *is a linear topological space,*
2. S *is a family of seminorms on* X *separating points in* X*,*
3. $Y \subseteq X$ *is compact in the weak topology* τ *generated by* S*,*
4. $f : Y \to Y$ *is a continuous mapping in the topology* τ *such that*

$$\forall s \in S \; \forall x, y \in Y \quad \lim_n \mathrm{s}(f^n(x) - f^n(y)) = 0$$

then f *has a unique fixed point in* Y*.*

Theorem

Let X be a linear topological space. Let U be a neighbourhood of the zero vector in X. We define Y as

$$Y := \{x^* \in X^* : |x(x^*)| \leq 1, \ \text{for each} \ x \in U\}$$

Let $f : Y \to Y$ be a weak-continuous mapping satisfying*

$$\forall z \in X \ \forall x, y \in Y \quad \lim_n z(f^n(x^*) - f^n(y^*)) = 0$$

Then f has a unique fixed point in Y.

`We use the dual notation:` $x(x^*) := x^*(x)$ `for functionals` x^* `which are members of` X^* `and elements` x `of the space` X.

Compact semigroups

Theorem

- *G is a Hausdorff compact topological monoid and*
- *$f : G \to G$ is a continuous mapping such that for each $x, y \in G$ and each neighbourhood V of the neutral element*

$$\exists z \in G \; \exists n \in \mathbb{N} \;\; f^n(x), f^n(y) \in zV$$

then f has a unique fixed point.

Theorem

If

1. *G is a first countable Hausdorff compact topological monoid,*
2. *$f : G \to G$ is a closed mapping such that for each $x, y \in G$ and each neighbourhood V of the neutral element*

$$\exists z \in G \; \exists n \in \mathbb{N} \;\; f^n(x), f^n(y) \in zV$$

then f has a unique fixed point.

Topological contraction

Definition

Let X be a T_1-topological space and $f : X \to X$.
We say that f is a topological contraction on X iff for every open cover $\mathcal{U}$ of X there are $U \in \mathcal{U}$ and $n \in \omega$ s.t. $f^n[X] \subseteq U$

Weak Čech completeness

Definition

Topological T_1 space X is weak Čech complete if

- exists $\{\mathcal{U}_i : i \in \omega\}$, $\mathcal{U}_i$ - open cover of X for $i \in \omega$,
- for every centered $\{F_m \in Clo(X) : m \in \omega\}$ s.t. $\forall i \in \omega\, \exists m \in \omega\, \exists U \in \mathcal{U}_i\; F_m \subseteq U$

then $\bigcap\{F_m \;\; m \in \omega\} \neq \emptyset$.

Theorem

If X is a T_1 weak Čech complete space and $f : X \to X$ is a closed topological contraction, then f has a unique fixed point.

Corollary

IF X is T_1 compact, $f : X \to X$ closed topological contraction, then f has a unique fixed point.

Example

Let (ω, τ) be T_1 topological space where

$$\tau = \{\emptyset\} \cup \{A \in \mathscr{P}(\omega) : A^c \text{ is finite } \}.$$

Then $\omega \ni n \mapsto f(n) = n + 1 \in \omega$ is a continuous, topological contraction without any fixed point, (f is not closed map !!!).

Lipschitz contraction is continuous but topological not neccessary.

Example

Let $X = \{1/n : n \in \mathbb{N}\} \cup \{0, 2, 3\}$ be endowed with the usual Euclidean metric from the real line. Let for $x \in X$:

$$f(x) := \begin{cases} 2 & \text{if} \quad x = 1/n, \\ 3 & \text{if} \quad x = 0, 2, 3. \end{cases}$$

The mapping f is a closed topological contraction because $f^2[X] = \{3\}$; it is closed because $f[X] = \{2, 3\}$; and it is not continuous because

$$f\left(\lim_n \frac{1}{n}\right) = f(0) = 3 \neq 2 = \lim_n f\left(\frac{1}{n}\right).$$

Here fixed point here is 3. Moreover, $f \subseteq X \times X$ is not closed set.

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Thank You