Algebraic sums, trees and meager sets in the Cantor space and in the Baire space

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Question

Let $P \subseteq \mathbb{R}$ be a perfect set, $M \in \mathcal{M}$. Can we find perfect $P' \subseteq P$ such that

$$M + \underbrace{P' + P' + \cdots + P'}_{n-times} \in \mathcal{M}?$$

Motivation

Let $\mathfrak{c}$ be regular. Then for every generalized Luzin set L and generalized Sierpiński set S we have $L + S \in s_0$.

- $L \subseteq \mathbb{R}$ ia a generalized Luzin set if $\forall M \in \mathcal{M}\ |M \cap L| < |L|$;
- $S \subseteq \mathbb{R}$ ia a generalized Sierpiński set if $\forall N \in \mathcal{N}\ |N \cap S| < |S|$.

M. Michalski, Sz. Żeberski, Some properties of $\mathcal{I}-$Luzin sets, Topology and its Applications 189 (2015), pp. 122–135.

Jan van Mill, usefull tip

- How do you prove it?
- You need to know what to do!

J. van Mill, Ryll-Nardzewski Lecture,
2nd Wrocław Logic Conference, 2024.

Basic meager set in 2^ω

Let $F \in \mathcal{M}$.

There is $x_F \in 2^\omega$ and $\{I_n : n \in \omega\}$ such that

$$F \subseteq \{x \in 2^\omega : (\forall^\infty n)(x \restriction I_n \neq x_F \restriction I_n)\}.$$

Bartoszyński T., Judah H., *Set theory: On the structure of the real line*, A K Peters. Ltd., 1995.

Definition

A tree $T \subseteq 2^{<\omega}$ is

- perfect if $(\forall \sigma \in T)(\exists \tau \supseteq \sigma)(\tau^\frown 0, \tau^\frown 1 \in T)$;
- a Silver tree if T is perfect and

$$(\exists x \in 2^\omega)(\exists A \in [\omega]^\omega)(\forall \sigma \in T)(\forall n \in \mathrm{dom}(\sigma))$$
$$(n \notin A \rightarrow \sigma(n) = x(n));$$

Silver tree case

If $F \in \mathcal{M}$ and T is a Silver tree, then there is a Silver tree $T' \subseteq T$ such that

$$F + \underbrace{[T'] + [T'] + \ldots + [T']}_{n\text{–}times} \in \mathcal{M}.$$

Proof.

Let $J_n = I_{2n} \cup I_{2n+1}$ and $A' \subseteq A$ such that $|A' \cap J_n| \leq 1$. Let

$$x_{F_n} = x_F + \underbrace{x_T + x_T + \cdots + x_T}_{n\text{–}times}$$

for each $n \in \omega$. For each $n \in \omega$ set

$$F_n = \{x \in 2^\omega : (\forall^\infty k)(x \restriction J_k \neq x_{F_n} \restriction J_k)\}$$

□

Definition

A tree $T \subseteq 2^{<\omega}$ is uniformly perfect if it is perfect and

$$(\forall \sigma, \tau \in T)((|\sigma| = |\tau|) \rightarrow (\sigma^\frown 0, \sigma^\frown 1 \in T \rightarrow \tau^\frown 0, \tau^\frown 1 \in T)).$$

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(Uniformly) perfect tree case

For a meager set $F \subseteq 2^{\omega}$ and a (uniformly) perfect tree T there is a (uniformly) perfect tree $T' \subseteq T$ such that for every $n \in \omega$

$$F + \underbrace{[T'] + [T'] + ... + [T']}_{n\text{–}times} \in \mathcal{M}.$$

Definition

A tree $T \subseteq 2^{<\omega}$ is a splitting tree if

$$(\forall \tau \in T)(\exists N \in \omega)(\forall n \geq N)(\forall i \in 2)$$
$$(\exists \tau' \in T \cap 2^{n+1})(\tau \subseteq \tau' \wedge \tau'(n) = i).$$

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Partial result for splitting trees

For a meager set $F \subseteq 2^{\omega}$ there is a splitting tree T such that

$$F + \underbrace{[T] + [T] + ... + [T]}_{n\text{–}times} \in \mathcal{M}.$$

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Partial result for splitting trees

For a meager set $F \subseteq 2^{\omega}$ there is a splitting tree T such that

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Problem

Let F be a meager subset of 2^{ω}. Is it true that for every splitting tree T there exists a splitting tree $T' \subseteq T$ such that

$$F + \underbrace{[T'] + [T'] + ... + [T']}_{\text{n-times}} \in \mathcal{M}?$$

The Baire space

$A \in \mathcal{M}_-$ if there is $x_A \in \mathbb{Z}^\omega$ and a partition $\{I_n : n \in \omega\}$ of ω into intervals such that

$$A \subseteq \{x \in \omega^\omega : (\forall^\infty n)(x \restriction I_n \neq x_A \restriction I_n)\}.$$

By nwd_- the ideal of sets generated by

$$\{x \in \omega^\omega : (\forall n)(x \restriction I_n \neq x_A \restriction I_n)\}.$$

Notice that $\mathcal{M}_-$ is a translation invariant σ-ideal with the basis of class F_σ. Also, $\mathcal{K}_\sigma \subsetneq \mathcal{M}_-$ and $\mathcal{M}_- \subseteq \mathcal{M}$.

Theorem

$\mathcal{M} \neq \mathcal{M}_-$.

What can we copy from the Cantor space?

- For every $F \in \mathcal{M}_-$ and every (uniformly) perfect tree $T \subseteq \mathbb{Z}^{<\omega}$ there is a (uniformly) perfect tree $T' \subseteq T$ such that

$$F + \underbrace{[T'] + [T'] + \cdots + [T']}_{n\text{–}times} \in \mathcal{M}_-.$$

- For every $F \in \mathcal{M}_-$ and every $\omega-$Silver tree $T \subseteq \mathbb{Z}^{<\omega}$ there is a $\omega-$Silver tree $T' \subseteq T$ such that

$$F + \underbrace{[T'] + [T'] + \cdots + [T']}_{n\text{–}times} \in \mathcal{M}_-.$$

Definition

A tree $T \subseteq \mathbb{Z}^{<\omega}$ is $\omega-$Silver, if there are $A \in [\omega]^\omega$ and x_T such that

$$T = \{\sigma \in \mathbb{Z}^{<\omega} : (\forall n \in \mathrm{dom}(\sigma) \setminus A)(\sigma(n) = x_T(n))\}.$$

Characterization of $\mathcal{M}$ in the Baire space

For every meager set $F \subseteq \mathbb{Z}^\omega$ there exists $f : \mathbb{Z}^{<\omega} \to \mathbb{Z}^{<\omega}$ such that

$$F \subseteq \{x \in \mathbb{Z}^\omega : (\forall^\infty \sigma \in \mathbb{Z}^{<\omega})(\sigma^\frown f(\sigma) \not\subseteq x)\}.$$

Theorem

For every $F \in \mathcal{M}$ and every (uniformly) perfect tree $T \subseteq \mathbb{Z}^{<\omega}$ there is a (uniformly) perfect tree $T' \subseteq T$ such that for each n

$$F + \underbrace{[T'] + [T'] + \cdots + [T']}_{n-times} \in \mathcal{M}.$$

Definition

We call a tree $T \subseteq \mathbb{Z}^{<\omega}$

- Miller, if $(\forall \sigma \in T)(\exists \tau \in T)(\sigma \subseteq \tau \wedge \tau \in \omega\text{-split}(T))$;
- Laver, if $(\exists \sigma \in T)(\forall \tau \in T)(\tau \subseteq \sigma \vee (\sigma \subseteq \tau \wedge \tau \in \omega\text{-split}(T)))$.

Laver tree case

There exists a set $A \in \mathcal{M}_-$ such that $A + [T] = \mathbb{Z}^\omega$ for each Laver tree T.

Proof.

Define

$$A = \{x \in \mathbb{Z}^\omega : (\forall^\infty n)(x(n) \neq 0)\}.$$

Let T be a Laver tree and let $\sigma_0 = \text{stem}(T)$. Let $z \in \mathbb{Z}^\omega$. We will find $x \in A$ and $y \in [T]$ satisfying $x + y = z$. □

Miller tree case

There is an nwd_{-} set F and a Miller tree T such that for any Miller tree $T' \subseteq T$

$$F + [T'] \notin \mathcal{M}.$$

Proof.

Let

$$F = \{x \in \mathbb{Z}^{\omega} : (\forall n)(x(n) \neq 0)\},$$

Fix a bijection $\alpha : \mathbb{Z}^{<\omega} \to \mathbb{Z}$ and let $\hat{\alpha} : \mathbb{Z}^{<\omega} \to \mathbb{Z}^{<\omega}$ be given by

$$\hat{\alpha}(\emptyset) = \emptyset,$$
$$\hat{\alpha}(\sigma^\frown i) = \hat{\alpha}(\sigma)^\frown \alpha(\sigma^\frown i).$$

Set $T = \mathrm{rng}(\hat{\alpha})$. Clearly T is a Miller tree. □

ω-Silver and meager

There is a nowhere dense set F such that for each $\omega-$Silver tree T we have $F + [T] \notin \mathcal{M}$.

Proof.

Wlog, $x_T = (0, 0, \dots)$, i.e.

$$[T] = \{x \in \mathbb{Z}^\omega : (\forall n \notin A)(x(n) = 0)\}.$$

. Let

$$F = \{x \in \mathbb{Z}^\omega : (\forall n \in \omega)(x \not\supseteq \sigma_n{}^\frown \underbrace{0 \dots 0}_{n-times})\},$$

where $(\sigma_n : n \in \omega)$ is an enumeration of $\mathbb{Z}^{<\omega}$. □

Thank you for your attention!

References

M. Michalski, R. Rałowski, Sz. Żeberski, On algebraic sums, trees and ideals in the Cantor space, https://arxiv.org/abs/2405.13775,

Ł. Mazurkiewicz, M. Michalski, R. Rałowski, Sz. Żeberski, On algebraic sums, trees and ideals in the Baire space, https://arxiv.org/abs/2409.17748.