Ideals on ω and Nikodym vs Grothendieck property of Boolean algebras

Tomasz Żuchowski

Mathematical Institute, University of Wrocław

Winter School in Abstract Analysis,
section Set Theory & Topology
Hejnice, 28.01.2025

joint work with Damian Sobota

A *measure* μ on a Boolean algebra $\mathcal{A}$ is a finitely additive function $\mu: \mathcal{A} \to \mathbb{R}$ of finite total variation, that is, A *measure* μ on a Boolean algebra $\mathcal{A}$ is a finitely additive function $\mu\colon \mathcal{A} \to \mathbb{R}$ of finite total variation, that is,

$$\|\mu\| = \sup\left\{|\mu(A)| + |\mu(B)| \colon A, B \in \mathcal{A}, A \wedge B = 0_{\mathcal{A}}\right\} < \infty.$$

The Nikodym property

A *measure* μ on a Boolean algebra $\mathcal{A}$ is a finitely additive function $\mu\colon \mathcal{A} \to \mathbb{R}$ of finite total variation, that is,

$$\|\mu\| = \sup\left\{|\mu(A)| + |\mu(B)| \colon A, B \in \mathcal{A}, A \wedge B = 0_{\mathcal{A}}\right\} < \infty.$$

A sequence of measures $\langle \mu_n \colon n \in \omega \rangle$ on $\mathcal{A}$ is

- *pointwise null* if $\mu_n(A) \to 0$ for every $A \in \mathcal{A}$,
- *uniformly bounded* if $\sup_n \|\mu_n\| < \infty$.

The Nikodym property

A *measure* μ on a Boolean algebra $\mathcal{A}$ is a finitely additive function $\mu\colon \mathcal{A} \to \mathbb{R}$ of finite total variation, that is,

$$\|\mu\| = \sup\left\{|\mu(A)| + |\mu(B)| \colon A, B \in \mathcal{A}, A \wedge B = 0_{\mathcal{A}}\right\} < \infty.$$

A sequence of measures $\langle \mu_n \colon n \in \omega \rangle$ on $\mathcal{A}$ is

- *pointwise null* if $\mu_n(A) \to 0$ for every $A \in \mathcal{A}$,
- *uniformly bounded* if $\sup_n \|\mu_n\| < \infty$.

Nikodym property of Boolean algebra

A Boolean algebra $\mathcal{A}$ has the *Nikodym property* if every pointwise null sequence of measures on $\mathcal{A}$ is uniformly bounded.

The Nikodym property - examples

Examples of Boolean algebras with the Nikodym property

- σ-complete algebras (Nikodym '30, Andô '61),

The Nikodym property - examples

Examples of Boolean algebras with the Nikodym property

- σ-complete algebras (Nikodym '30, Andô '61),
- algebras with Subsequential Completeness Property (Haydon '81),

The Nikodym property - examples

Examples of Boolean algebras with the Nikodym property

- σ-complete algebras (Nikodym '30, Andô '61),
- algebras with Subsequential Completeness Property (Haydon '81),
- the algebra of Jordan measurable subsets of $[0, 1]$ (Schachermayer '82).

The Nikodym property - examples

Examples of Boolean algebras with the Nikodym property

- σ-complete algebras (Nikodym '30, Andô '61),
- algebras with Subsequential Completeness Property (Haydon '81),
- the algebra of Jordan measurable subsets of $[0, 1]$ (Schachermayer '82).

However, if the Stone space $St(\mathcal{A})$ of ultrafilters on $\mathcal{A}$ contains a non-trivial convergent sequence, then $\mathcal{A}$ does not have the Nikodym property:

The Nikodym property - examples

Examples of Boolean algebras with the Nikodym property

- σ-complete algebras (Nikodym '30, Andô '61),
- algebras with Subsequential Completeness Property (Haydon '81),
- the algebra of Jordan measurable subsets of $[0,1]$ (Schachermayer '82).

However, if the Stone space $St(\mathcal{A})$ of ultrafilters on $\mathcal{A}$ contains a non-trivial convergent sequence, then $\mathcal{A}$ does not have the Nikodym property:

if $x_n \to x$, then consider the sequence of measures $\mu_n = n(\delta_{x_n} - \delta_x)$.

N_F spaces

N_F spaces

F - a free filter on ω

$N_F = \omega \cup \{p_F\}$, where $p_F \notin \omega$, with the following topology:

- every point of ω is isolated in N_F,
- U is an open neighborhood of p_F iff $A \cup \{p_F\}$ for some $A \in F$.

N_F spaces

F - a free filter on ω

$N_F = \omega \cup \{p_F\}$, where $p_F \notin \omega$, with the following topology:

- every point of ω is isolated in N_F,
- U is an open neighborhood of p_F iff $A \cup \{p_F\}$ for some $A \in F$.

Trivial example

N_{Fr} is homeomorphic to a convergent sequence (together with its limit), where by Fr we denote the Fréchet filter on ω.

N_F spaces

N_F spaces

F - a free filter on ω

$N_F = \omega \cup \{p_F\}$, where $p_F \notin \omega$, with the following topology:

- every point of ω is isolated in N_F,
- U is an open neighborhood of p_F iff $A \cup \{p_F\}$ for some $A \in F$.

Trivial example

N_{Fr} is homeomorphic to a convergent sequence (together with its limit), where by Fr we denote the Fréchet filter on ω.

Question

For which filters F on ω, if N_F homeomorphically embeds into the Stone space $St(\mathcal{A})$ of a Boolean algera $\mathcal{A}$, then $\mathcal{A}$ does not have the Nikodym property?

Class $\mathcal{AN}$ of ideals

A Borel measure μ on a topological space X is *finitely supported* if $\mu = \sum_{i=1}^{n} \alpha_i \delta_{x_i}$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and distinct $x_1, \ldots, x_n \in X$.

Class $\mathcal{AN}$ of ideals

A Borel measure μ on a topological space X is *finitely supported* if $\mu = \sum_{i=1}^{n} \alpha_i \delta_{x_i}$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and distinct $x_1, \ldots, x_n \in X$.

In this case we have $\|\mu\| = \sum_{i=1}^{n} |\alpha_i|$ and *supp*$(\mu_n) = \{x_1, \ldots, x_n\}$.

Class $\mathcal{AN}$ of ideals

A Borel measure μ on a topological space X is *finitely supported* if $\mu = \sum_{i=1}^{n} \alpha_i \delta_{x_i}$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and distinct $x_1, \ldots, x_n \in X$.

In this case we have $\|\mu\| = \sum_{i=1}^{n} |\alpha_i|$ and $supp(\mu_n) = \{x_1, \ldots, x_n\}$.

Class $\mathcal{AN}$

$\mathcal{I}$ - ideal on ω
$\mathcal{I} \in \mathcal{AN}$ if there is no sequence of finitely supported measures $\langle \mu_n : n \in \omega \rangle$ on $N_{\mathcal{I}^*}$ such that

Class $\mathcal{AN}$ of ideals

A Borel measure μ on a topological space X is *finitely supported* if $\mu = \sum_{i=1}^{n} \alpha_i \delta_{x_i}$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and distinct $x_1, \ldots, x_n \in X$.

In this case we have $\|\mu\| = \sum_{i=1}^{n} |\alpha_i|$ and $supp(\mu_n) = \{x_1, \ldots, x_n\}$.

Class $\mathcal{AN}$

$\mathcal{I}$ - ideal on ω
$\mathcal{I} \in \mathcal{AN}$ if there is no sequence of finitely supported measures $\langle \mu_n : n \in \omega \rangle$ on $N_{\mathcal{I}^*}$ such that

- $\|\mu_n\| \to \infty$,

Class $\mathcal{AN}$ of ideals

A Borel measure μ on a topological space X is *finitely supported* if $\mu = \sum_{i=1}^{n} \alpha_i \delta_{x_i}$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and distinct $x_1, \ldots, x_n \in X$.

In this case we have $\|\mu\| = \sum_{i=1}^{n} |\alpha_i|$ and $supp(\mu_n) = \{x_1, \ldots, x_n\}$.

Class $\mathcal{AN}$

$\mathcal{I}$ - ideal on ω
$\mathcal{I} \in \mathcal{AN}$ if there is no sequence of finitely supported measures $\langle \mu_n : n \in \omega \rangle$ on $N_{\mathcal{I}^*}$ such that

- $\|\mu_n\| \to \infty$,
- $\mu_n(A) \to 0$ for every $A \in \mathrm{Clopen}(N_{\mathcal{I}^*})$.

Class $\mathcal{AN}$ of ideals

A Borel measure μ on a topological space X is *finitely supported* if $\mu = \sum_{i=1}^{n} \alpha_i \delta_{x_i}$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and distinct $x_1, \ldots, x_n \in X$.

In this case we have $\|\mu\| = \sum_{i=1}^{n} |\alpha_i|$ and $supp(\mu_n) = \{x_1, \ldots, x_n\}$.

Class $\mathcal{AN}$

$\mathcal{I}$ - ideal on ω

$\mathcal{I} \in \mathcal{AN}$ if there is no sequence of finitely supported measures $\langle \mu_n : n \in \omega \rangle$ on $N_{\mathcal{I}^*}$ such that

- $\|\mu_n\| \to \infty$,
- $\mu_n(A) \to 0$ for every $A \in \text{Clopen}(N_{\mathcal{I}^*})$.

Theorem (Ż.)

If $\mathcal{I} \in \mathcal{AN}$ and $N_{\mathcal{I}^*}$ homeomorphically embeds into $St(\mathcal{A})$, then $\mathcal{A}$ does not have the Nikodym property.

Definition

$\varphi\colon \mathcal{P}(\omega) \to [0,\infty]$ is an *lsc submeasure* if

- $\varphi(\emptyset) = 0$ and $\varphi(\{n\}) < \infty$ for every $n \in \omega$,
- $\varphi(X) \leq \varphi(Y)$ whenever $X \subseteq Y$,
- $\varphi(X \cup Y) \leq \varphi(X) + \varphi(Y)$ for every $X, Y \subseteq \omega$,
- (lsc) $\varphi(A) = \lim_{n\to\infty} \varphi(A \cap [0,n])$ for every $A \subseteq \omega$

Submeasures and ideals on ω

Definition

$\varphi\colon \mathcal{P}(\omega) \to [0,\infty]$ is an *lsc submeasure* if

- $\varphi(\varnothing) = 0$ and $\varphi(\{n\}) < \infty$ for every $n \in \omega$,
- $\varphi(X) \leq \varphi(Y)$ whenever $X \subseteq Y$,
- $\varphi(X \cup Y) \leq \varphi(X) + \varphi(Y)$ for every $X, Y \subseteq \omega$,
- (lsc) $\varphi(A) = \lim_{n\to\infty} \varphi(A \cap [0,n])$ for every $A \subseteq \omega$

Ideals associated with an lsc submeasure φ on ω

$Exh(\varphi) = \{A \subseteq \omega\colon \lim_{n\to\infty} \varphi(A \setminus [0,n]) = 0\}$ – an $F_{\sigma\delta}$ P-ideal
$Fin(\varphi) = \{A \subseteq \omega\colon \varphi(A) < \infty\}$ – an F_σ ideal

Submeasures and ideals on ω

Definition

$\varphi: \mathcal{P}(\omega) \to [0, \infty]$ is an *lsc submeasure* if

- $\varphi(\emptyset) = 0$ and $\varphi(\{n\}) < \infty$ for every $n \in \omega$,
- $\varphi(X) \leq \varphi(Y)$ whenever $X \subseteq Y$,
- $\varphi(X \cup Y) \leq \varphi(X) + \varphi(Y)$ for every $X, Y \subseteq \omega$,
- (lsc) $\varphi(A) = \lim_{n\to\infty} \varphi(A \cap [0, n])$ for every $A \subseteq \omega$

Ideals associated with an lsc submeasure φ on ω

$Exh(\varphi) = \left\{A \subseteq \omega \colon \lim_{n\to\infty} \varphi(A \setminus [0, n]) = 0\right\}$ – an $F_{\sigma\delta}$ P-ideal
$Fin(\varphi) = \left\{A \subseteq \omega \colon \varphi(A) < \infty\right\}$ – an F_σ ideal

Summable ideals

For every $f : \omega \to \mathbb{R}_+$ such that $\sum_{n\in\omega} f(n) = \infty$, let
$\mathcal{I}_f = \left\{A \subseteq \omega : \sum_{n\in A} f(n) < \infty\right\}$

Submeasures and ideals on ω

Definition

$\varphi\colon \mathcal{P}(\omega) \to [0,\infty]$ is an *lsc submeasure* if

- $\varphi(\varnothing) = 0$ and $\varphi(\{n\}) < \infty$ for every $n \in \omega$,
- $\varphi(X) \leq \varphi(Y)$ whenever $X \subseteq Y$,
- $\varphi(X \cup Y) \leq \varphi(X) + \varphi(Y)$ for every $X, Y \subseteq \omega$,
- (lsc) $\varphi(A) = \lim_{n\to\infty} \varphi(A \cap [0,n])$ for every $A \subseteq \omega$

Ideals associated with an lsc submeasure φ on ω

$Exh(\varphi) = \left\{A \subseteq \omega\colon \lim_{n\to\infty} \varphi(A \setminus [0,n]) = 0\right\}$ – an $F_{\sigma\delta}$ P-ideal
$Fin(\varphi) = \left\{A \subseteq \omega\colon \varphi(A) < \infty\right\}$ – an F_σ ideal

Summable ideals

For every $f\colon \omega \to \mathbb{R}_+$ such that $\sum_{n\in\omega} f(n) = \infty$, let
$\mathcal{I}_f = \left\{A \subseteq \omega\colon \sum_{n\in A} f(n) < \infty\right\} = Exh(\mu_f) = Fin(\mu_f)$,
where $\mu_f(A) = \sum_{n\in A} f(n)$ – a non-negative measure on ω.

Special types of submeasures and ideals

For submeasures φ, ψ we write $\psi \leq \varphi$ if $\psi(A) \leq \varphi(A)$ for all $A \subseteq \omega$.

Special types of submeasures and ideals

For submeasures φ, ψ we write $\psi \leq \varphi$ if $\psi(A) \leq \varphi(A)$ for all $A \subseteq \omega$.

Non-pathological submeasure

An lsc submeasure φ is *non-pathological* if for every $A \subseteq \omega$ we have: $\varphi(A) = sup\{\mu(A) \colon \mu$ is a non-negative measure on ω s.t. $\mu \leq \varphi\}$.

$\mathcal{I}$ is a non-pathological ideal if $\mathcal{I} = Exh(\varphi)$ for some non-pathological submeasure.

Special types of submeasures and ideals

For submeasures φ, ψ we write $\psi \leq \varphi$ if $\psi(A) \leq \varphi(A)$ for all $A \subseteq \omega$.

Non-pathological submeasure

An lsc submeasure φ is *non-pathological* if for every $A \subseteq \omega$ we have: $\varphi(A) = sup\{\mu(A) : \mu$ is a non-negative measure on ω s.t. $\mu \leq \varphi\}$.

$\mathcal{I}$ is a non-pathological ideal if $\mathcal{I} = Exh(\varphi)$ for some non-pathological submeasure.

Density submeasure

A submeasure φ is a *density submeasure* if there exists a sequence $\langle \mu_n : n \in \omega \rangle$ of non-negative measures on ω with finite disjoint supports such that $\varphi = \sup_{n \in \omega} \mu_n$.

Special types of submeasures and ideals

For submeasures φ, ψ we write $\psi \leq \varphi$ if $\psi(A) \leq \varphi(A)$ for all $A \subseteq \omega$.

Non-pathological submeasure

An lsc submeasure φ is *non-pathological* if for every $A \subseteq \omega$ we have: $\varphi(A) = sup\{\mu(A) : \mu$ is a non-negative measure on ω s.t. $\mu \leq \varphi\}$.

$\mathcal{I}$ is a non-pathological ideal if $\mathcal{I} = Exh(\varphi)$ for some non-pathological submeasure.

Density submeasure

A submeasure φ is a *density submeasure* if there exists a sequence $\langle \mu_n : n \in \omega \rangle$ of non-negative measures on ω with finite disjoint supports such that $\varphi = \sup_{n \in \omega} \mu_n$.

The canonical example is an *asymptotic density* on ω defined by:

$$\varphi_d(A) = \sup_{n \in \omega} \left| A \cap [2^n, 2^{n+1}) \right| / 2^n.$$

Theorem (Ż.)

Let $\mathcal{I}$ be an ideal on ω. Then, the following are equivalent:

Theorem (Ż.)

Let $\mathcal{I}$ be an ideal on ω. Then, the following are equivalent:

- $\mathcal{I} \in \mathcal{AN}$;

Theorem (Ż.)

Let $\mathcal{I}$ be an ideal on ω. Then, the following are equivalent:

- $\mathcal{I} \in \mathcal{AN}$;
- *there is a density submeasure φ on ω such that $\varphi(\omega) = \infty$ and $\mathcal{I} \subseteq Exh(\varphi)$;*

Theorem (Ż.)

Let $\mathcal{I}$ be an ideal on ω. Then, the following are equivalent:

- $\mathcal{I} \in \mathcal{AN}$;
- *there is a density submeasure φ on ω such that $\varphi(\omega) = \infty$ and $\mathcal{I} \subseteq Exh(\varphi)$;*
- *there is a non-pathological lsc submeasure φ on ω such that $\varphi(\omega) = \infty$ and $\mathcal{I} \subseteq Exh(\varphi)$.*

Theorem (Ż.)

Let $\mathcal{I}$ be an ideal on ω. Then, the following are equivalent:

- $\mathcal{I} \in \mathcal{AN}$;
- *there is a density submeasure φ on ω such that $\varphi(\omega) = \infty$ and $\mathcal{I} \subseteq Exh(\varphi)$;*
- *there is a non-pathological lsc submeasure φ on ω such that $\varphi(\omega) = \infty$ and $\mathcal{I} \subseteq Exh(\varphi)$.*

Theorem (folklore)

For every density submeasure φ on ω such that $\varphi(\omega) = \infty$ there is a summable ideal containing $Exh(\varphi)$.

Theorem (Ż.)

Let $\mathcal{I}$ be an ideal on ω. Then, the following are equivalent:

- $\mathcal{I} \in \mathcal{AN}$;
- *there is a density submeasure φ on ω such that $\varphi(\omega) = \infty$ and $\mathcal{I} \subseteq Exh(\varphi)$;*
- *there is a non-pathological lsc submeasure φ on ω such that $\varphi(\omega) = \infty$ and $\mathcal{I} \subseteq Exh(\varphi)$.*

Theorem (folklore)

For every density submeasure φ on ω such that $\varphi(\omega) = \infty$ there is a summable ideal containing $Exh(\varphi)$.

Corollary

$\mathcal{I} \in \mathcal{AN}$ if and only if $\mathcal{I}$ is contained in a summable ideal.

Totally bounded ideals

Definition (Hernández-Hernández, Hrušák)

An analytic P-ideal $\mathcal{I}$ on ω is *totally bounded* if whenever φ is an lsc submeasure on ω for which $\mathcal{I} = Exh(\varphi)$, then $\varphi(\omega) < \infty$

Totally bounded ideals

Definition (Hernández-Hernández, Hrušák)

An analytic P-ideal $\mathcal{I}$ on ω is *totally bounded* if whenever φ is an lsc submeasure on ω for which $\mathcal{I} = Exh(\varphi)$, then $\varphi(\omega) < \infty$

Lemma (Sobota, Ż.)

For every lsc submeasure φ on ω such that $\varphi(\omega) = \infty$ there exists an lsc submeasure ψ on ω satisfying $\psi(\omega) = \infty$ and $Fin(\varphi) \subseteq Exh(\psi)$.

Totally bounded ideals

Definition (Hernández-Hernández, Hrušák)

An analytic P-ideal $\mathcal{I}$ on ω is *totally bounded* if whenever φ is an lsc submeasure on ω for which $\mathcal{I} = Exh(\varphi)$, then $\varphi(\omega) < \infty$

Lemma (Sobota, Ż.)

For every lsc submeasure φ on ω such that $\varphi(\omega) = \infty$ there exists an lsc submeasure ψ on ω satisfying $\psi(\omega) = \infty$ and $Fin(\varphi) \subseteq Exh(\psi)$.

Theorem (Sobota, Ż.)

An analytic P-ideal $\mathcal{I}$ on ω is totally bounded if and only if $\mathcal{I}$ is not contained in an F_σ ideal.

Totally bounded ideals and the Nikodym property

Theorem (Ż.)

For a density submeasure φ and an ideal $\mathcal{I} = Exh(\varphi)$ on ω we have $\mathcal{I} \in \mathcal{AN}$ if and only if $\mathcal{I}$ is not totally bounded.

Totally bounded ideals and the Nikodym property

Theorem (Ż.)

For a density submeasure φ and an ideal $\mathcal{I} = Exh(\varphi)$ on ω we have $\mathcal{I} \in \mathcal{AN}$ if and only if $\mathcal{I}$ is not totally bounded.

Definition

An ideal $\mathcal{I}$ on ω is a *hypergraph ideal* if, for some $\langle G_n : n \in \omega \rangle$ – finite non-empty disjoint subsets of ω and $H_n \subseteq [G_n]^{<\omega}$, we have

$$\mathcal{I} = \left\{ A \subseteq \omega : \lim_{n\to\infty} \sup_{e \in H_n} \frac{|A \cap e|}{|e|} = 0 \right\}.$$

Totally bounded ideals and the Nikodym property

Theorem (Ż.)

For a density submeasure φ and an ideal $\mathcal{I} = Exh(\varphi)$ on ω we have $\mathcal{I} \in \mathcal{AN}$ if and only if $\mathcal{I}$ is not totally bounded.

Definition

An ideal $\mathcal{I}$ on ω is a *hypergraph ideal* if, for some $\langle G_n : n \in \omega \rangle$ – finite non-empty disjoint subsets of ω and $H_n \subseteq [G_n]^{<\omega}$, we have

$$\mathcal{I} = \left\{ A \subseteq \omega : \lim_{n \to \infty} \sup_{e \in H_n} \frac{|A \cap e|}{|e|} = 0 \right\}.$$

Theorem (Sobota, Ż.)

There is a family $\{\mathcal{I}_\alpha : \alpha < \mathfrak{c}\}$ of pairwise non-isomorphic hypergraph ideals (in particular: non-pathological ideals) which are outside the class $\mathcal{AN}$ and are not totally bounded.

Comparison with the Grothendieck property

Grothendieck property of Boolean algebra

A Boolean algebra $\mathcal{A}$ has the *Grothendieck property* if the Banach space $C\big(St(\mathcal{A})\big)$ is a Grothendieck space.

Comparison with the Grothendieck property

Grothendieck property of Boolean algebra

A Boolean algebra $\mathcal{A}$ has the *Grothendieck property* if the Banach space $C\big(St(\mathcal{A})\big)$ is a Grothendieck space.

Not easy to distinguish with the Nikodym property!

Comparison with the Grothendieck property

Grothendieck property of Boolean algebra

A Boolean algebra $\mathcal{A}$ has the *Grothendieck property* if the Banach space $C\big(St(\mathcal{A})\big)$ is a Grothendieck space.

Not easy to distinguish with the Nikodym property!

Nikodym property vs Grothendieck property

- (Schachermayer, 1982) the algebra of Jordan measurable subsets of $[0,1]$ has (N) but not (G)

Comparison with the Grothendieck property

Grothendieck property of Boolean algebra

A Boolean algebra $\mathcal{A}$ has the *Grothendieck property* if the Banach space $C\big(St(\mathcal{A})\big)$ is a Grothendieck space.

Not easy to distinguish with the Nikodym property!

Nikodym property vs Grothendieck property

- (Schachermayer, 1982) the algebra of Jordan measurable subsets of $[0,1]$ has (N) but not (G)
- (Talagrand, 1984) example under CH of the algebra with (G) but without (N)

Comparison with the Grothendieck property

Grothendieck property of Boolean algebra

A Boolean algebra $\mathcal{A}$ has the *Grothendieck property* if the Banach space $C\big(St(\mathcal{A})\big)$ is a Grothendieck space.

Not easy to distinguish with the Nikodym property!

Nikodym property vs Grothendieck property

- (Schachermayer, 1982) the algebra of Jordan measurable subsets of $[0,1]$ has (N) but not (G)
- (Talagrand, 1984) example under CH of the algebra with (G) but without (N)
- (Sobota & Zdomskyy, 2023) example under MA of the algebra with (G) but without (N)
- (Głodkowski & Widz, 2024) forcing construction of the algebra with (G) but without (N)

Comparison with the BJN property

Theorem (Marciszewski, Sobota)

Let F be a filter on ω and $\mathcal{A}$ a Boolean algebra. Then,

- *N_F has the BJN property*

Comparison with the BJN property

Theorem (Marciszewski, Sobota)

Let F be a filter on ω and $\mathcal{A}$ a Boolean algebra. Then,

- *N_F has the BJN property if and only if there is a non-pathological submeasure φ on ω such that $F \subseteq Exh(\varphi)^*$;*

Theorem (Marciszewski, Sobota)

Let F be a filter on ω and $\mathcal{A}$ a Boolean algebra. Then,

- *N_F has the BJN property if and only if there is a non-pathological submeasure φ on ω such that $F \subsetneq Exh(\varphi)^*$;*
- *if N_F has the BJN property and embeds into $St(\mathcal{A})$, then $\mathcal{A}$ does not have the Grothendieck property.*

Definition

$\mathcal{A}_F = \{A \subseteq \omega \colon A \in F \vee A^c \in F\}$

Boolean algebras associated to filters

Definition

$\mathcal{A}_F = \{A \subseteq \omega \colon A \in F \vee A^c \in F\}$

Theorem (Sobota, Ż.)

If $\mathcal{A}_F$ does not have the Nikodym property, then it neither has the Grothendieck property.

Boolean algebras associated to filters

Definition

$\mathcal{A}_F = \{A \subseteq \omega \colon A \in F \vee A^c \in F\}$

Theorem (Sobota, Ż.)

If $\mathcal{A}_F$ does not have the Nikodym property, then it neither has the Grothendieck property.

Theorem (Sobota, Ż.)

The following are equivalent:

1. *$\mathcal{A}_F$ has the Nikodym property;*

Boolean algebras associated to filters

Definition

$\mathcal{A}_F = \{A \subseteq \omega \colon A \in F \vee A^c \in F\}$

Theorem (Sobota, Ż.)

If $\mathcal{A}_F$ does not have the Nikodym property, then it neither has the Grothendieck property.

Theorem (Sobota, Ż.)

The following are equivalent:

1. *$\mathcal{A}_F$ has the Nikodym property;*
2. *$\mathcal{A}_F/Fin$ has the Nikodym property and $F^* \notin \mathcal{AN}$.*

Large families of Boolean algebras associated to filters

Theorem (Sobota, Ż.)

For every non-pathological ideal $\mathcal{I}$ such that $\mathcal{I} \notin \mathcal{AN}$, the algebra $\mathcal{A}_{\mathcal{I}^*}$ has the Nikodym property and does not have the Grothendieck property.

Large families of Boolean algebras associated to filters

Theorem (Sobota, Ż.)

For every non-pathological ideal $\mathcal{I}$ such that $\mathcal{I} \notin \mathcal{AN}$, the algebra $\mathcal{A}_{\mathcal{I}^*}$ has the Nikodym property and does not have the Grothendieck property.

Corollary (Sobota, Ż.)

There are $\mathfrak{c}$ non-isomorphic Boolean algebras with (N) and without (G), of the form $\mathcal{A}_F$ where F is a Borel filter on ω.

Large families of Boolean algebras associated to filters

Theorem (Sobota, Ż.)

For every non-pathological ideal $\mathcal{I}$ such that $\mathcal{I} \notin \mathcal{AN}$, the algebra $\mathcal{A}_{\mathcal{I}^*}$ has the Nikodym property and does not have the Grothendieck property.

Corollary (Sobota, Ż.)

There are $\mathfrak{c}$ non-isomorphic Boolean algebras with (N) and without (G), of the form $\mathcal{A}_F$ where F is a Borel filter on ω.

Theorem (Sobota, Ż.)

For every non-principal ultrafilter $\mathcal{U}$ on ω, the algebra $\mathcal{A}_{\mathcal{U}\oplus\mathcal{Z}^*}$ has the Nikodym property and does not have the Grothendieck property, and so there are $2^{\mathfrak{c}}$ such non-isomorphic algebras.

The end

Thank You for Your attention :)

The end

Thank You for Your attention :)
But...

Advertisement

THE ROAMING LOGIC CONFERENCE
9th May - 11th May 2025
in Warsaw, Poland
`https://sites.google.com/uw.edu.pl/the-roaming-logic-conference`