# Local reflections of choice

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# The axiom of choice

f $\emptyset \notin X$ then $X$ has a *choice function*: $f: X \to \bigcup X$ such that $f(x) \in x$ for all $x \in X$. This has many desirable consequences that we can take as weakenings of AC:

1. $\mathrm{AC}_A(B)$: If $X = \{x_a \mid a \in A\} \subseteq \mathscr{P}(B)$ and $\emptyset \notin X$ then $X$ has a choice function.
2. $\mathrm{AC}_A$: If $|X| = |A|$ and $\emptyset \notin X$ then $X$ has a choice function.
3. The *principle of dependent choices* DC: If $T$ is a tree of infinite height then $T$ has a maximal node or a branch.
4. The *partition principle* PP: If there is a surjection $Y \to X$ then there is an injection $X \to Y$.

$|X| \leqslant |Y|$ means there is an injection $X \to Y$.
$|X| \leqslant^* |Y|$ means there is a surjection $Y \to X$, or $X = \emptyset$.

# Independence from ZFC

**Question (scheme)**

Let $\varphi$ be a theorem of ZFC. Is $\varphi$ a theorem of ZF?

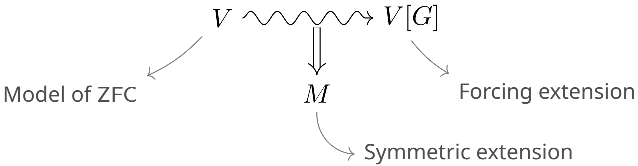
**Example**

In *Cohen's first model* (Cohen, 60s) there is an infinite set $A$ of real numbers such that $|\omega| \leqslant^* |A|$ but $|\omega| \nleqslant |A|$. Therefore, none of $\mathrm{AC}_\omega$, DC, or PP are theorems of ZF.

Cohen used (an early version of) *symmetric extensions*, which are still very powerful for independence proofs.

# Symmetric extensions

## Approximately




In the case of Cohen's first model, we add a set $A$ of $\omega$-many Cohen reals but 'forget' the enumeration so, in $M$, $A$ is *Dedekind-finite*.

# Symmetric extensions

## Exactly

A $\mathbb{P}$-name is a set $\dot{x}$ such that every $a \in \dot{x}$ is of the form $\langle p, \dot{y}\rangle$, where $p \in \mathbb{P}$ and $\dot{y}$ is a $\mathbb{P}$-name.

For $\pi \in \operatorname{Aut}(\mathbb{P})$, define

$$\pi\dot{x} = \{\langle \pi p, \pi\dot{y}\rangle \mid \langle p, \dot{y}\rangle \in \dot{x}\}.$$

A **normal filter of subgroups** of $\mathscr{G}$ is a set $\mathscr{F}$ of subgroups of $\mathscr{G}$ such that:

- If $H \in \mathscr{F}$ and $H \leqslant H' \leqslant \mathscr{G}$ then $H' \in \mathscr{F}$;
- if $H, H' \in \mathscr{F}$ then $H \cap H' \in \mathscr{F}$; and
- if $\pi \in \mathscr{G}$ and $H \in \mathscr{F}$ then $\pi H \pi^{-1} \in \mathscr{F}$.

A **symmetric system** is a triple $\mathscr{S} = \langle \mathbb{P}, \mathscr{G}, \mathscr{F}\rangle$ where $\mathscr{G} \leqslant \operatorname{Aut}(\mathbb{P})$ and $\mathscr{F}$ is a normal filter of subgroups of $\mathscr{G}$.

# Symmetric extensions

## Exactly

A **symmetric system** is a triple $\mathscr{S} = \langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle$ where $\mathscr{G} \leqslant \mathrm{Aut}(\mathbb{P})$ and $\mathscr{F}$ is a normal filter of subgroups of $\mathscr{G}$.

An $\mathscr{S}$**-name** is a $\mathbb{P}$-name such that $\{\pi \in \mathscr{G} \mid \pi\dot{x} = \dot{x}\} \in \mathscr{F}$, and this holds hereditarily for all names appearing in $\dot{x}$.

For a $V$-generic filter $G \subseteq \mathbb{P}$, we build the **symmetric extension**

$$V[G]_{\mathscr{S}} = \{\dot{x}^G \mid \dot{x} \text{ is an } \mathscr{S}\text{-name}\}.$$

**Fact**

$V[G]_{\mathscr{S}} \subseteq V[G]$ is a transitive model of ZF.

# Destruction is easy

Destroying a choice principle is generally easier than proving that it has not been destroyed.

- If we want to violate $\mathrm{AC}_\omega$ then we 'just' have to add a countable set $A$ with no choice function.
- If we want to check that $\mathrm{AC}_\omega$ holds in a model then we have to 'check' every countable set to see if it has a choice function.

**Example**

In Cohen's first model, every set can be linearly ordered, but building a bespoke linear order for every set is hard. Instead, we use its status as a symmetric extension.

# Local reflections of choice

## Theorem (Blass/Usuba, [Bla79; Usu21])

*Let* SVC$(S)$ *be the statement "for all* $X$ *there is* $\eta \in \mathrm{Ord}$ *such that* $|X| \leqslant^* |S \times \eta|$*".* $M$ *is a symmetric extension** *if and only if it is a model of* SVC $\equiv (\exists S)$SVC$(S)$.

If $M \vDash$ SVC$(S)$ and a choice principle fails, the failure is usually 'because of $S$'. That is, the failure has a *local reflection*.

## Example

- If $\mathsf{AC}_\omega$ fails then there is a countable set $X \subseteq \mathscr{P}(S)$ with no choice function.
- If DC fails then there is a subtree of $S^{<\omega}$ with no maximal nodes or cofinal branches.

# Local reflections of choice

SVC$(S)$ means "for all $X$ there is $\eta \in \mathrm{Ord}$ such that $|X| \leqslant^* |S \times \eta|$".

## Proposition (R.S.)

Assume SVC$(S)$. For all sets $X$, the following are equivalent:

1. AC$_X$; and
2. AC$_X(S)$.

## Proof.

Consider $A = \{A_y \mid y \in X\} \not\ni \emptyset$, and let $f \colon S \times \eta \to \bigcup A$ be a surjection. Let $S_y = \{t \in S \mid (\exists \beta < \eta) f(t, \beta) \in A_y\}$. By AC$_X(S)$, $\{S_y \mid y \in X\}$ has a choice function $c \colon X \to S$. Let $d(y) = f(c(y), \beta_y)$, where $\beta_y$ is least such that $f(c(y), \beta_y) \in A_y$. □

# Local reflections of choice

## Proposition (R.S.)

Assume SVC$(S)$. AC$_\omega$ is equivalent to AC$_\omega(S)$.

## Corollary

*Assume* SVC$(S)$. *If $S$ is infinite then $|\omega| \leqslant^* |S|$.*

## Proof.

If $|\omega| \nleqslant^* |S|$, then if $A = \{A_n \mid n < \omega\} \subseteq \mathscr{P}(S)$, for $t \in \bigcup A$ let $g(t) = \min\{n < \omega \mid t \in A_n\}$. Then $F = g``\bigcup A$ is finite, so there is a choice function $c \colon F \to S$. So $d(n) = c(\min g``A_n)$ is a choice function for $A$.

Therefore, AC$_\omega(S)$ holds, and so AC$_\omega$ holds.

However, AC$_\omega$ implies that if $X$ is infinite then $|\omega| \leqslant |X|$. □

# Local reflections of choice

Cohen's first model is a model of SVC$([A]^{<\omega})$, where $A \subseteq \mathbb{R}$ is such that $|\omega| \nleq |A|$. Since AC$_\omega$ fails, this is witnessed 'close to' $[A]^{<\omega}$. In fact, $\{[A]^n \mid n < \omega\}$ has no choice function.

If $c\colon \omega \to [A]^{<\omega}$ is a choice function then, since $A \subseteq \mathbb{R}$, there is a definable well-order on each $c(n)$, so we can well-order $c``\omega$ lexicographically and obtain an injection $\omega \to A$.

# SVC$^+$

(This frame added post-conference)

## Definition

SVC$^+(S)$ is *injective* SVC: For all $X$ there is $\eta \in \mathrm{Ord}$ such that $|X| \leqslant |S \times \eta|$.

Note that SVC$^+(S) \implies$ SVC$(S) \implies$ SVC$^+(\mathscr{P}(S))$.

# Local reflections of choice

Assume SVC$(S)$ and SVC$^+(T)$.

| | Consequence of AC | Local reflection |
|---|---|---|
| (Blass) | AC | $S$ can be well-ordered |
| | AC$_X$ | AC$_X(S)$ |
| | DC$_\lambda$ | DC$_\lambda$ for subtrees of $S^{<\lambda}$ |
| | Countable union theorem | $\mathrm{cf}(\omega_1) = \omega_1$ and $[T]^\omega$ is $\sigma$-closed |
| (Pincus) | BPI | There is a fine ultrafilter on $[S]^{<\omega}$ |
| | PP | AC$_{\mathrm{WO}}$ and PP $\upharpoonright T$: For all $X, Y \subseteq T$, if $\lvert X\rvert \leqslant^* \lvert Y\rvert$ then $\lvert X\rvert \leqslant \lvert Y\rvert$ |
| (Karagila–Schilhan) | KWP$_\alpha$ | $\lvert T\rvert \leqslant \lvert \mathscr{P}^\alpha(\mathrm{Ord})\rvert$ |
| (Karagila–Schilhan) | KWP$^*_\alpha$ | $\lvert S\rvert \leqslant^* \lvert \mathscr{P}^\alpha(\mathrm{Ord})\rvert$ |

# Questions

- Does PP imply AC?
- Does $\mathsf{SVC}^+(S) \wedge \mathsf{PP} \restriction S$ imply $\mathsf{AC}_{\mathsf{WO}}$? I.e., does $\mathsf{SVC}^+(S) \wedge \mathsf{PP} \restriction S$ imply $\mathsf{AC}_{\mathsf{WO}}$ on its own?
- Does $\mathrm{cf}(\omega_1) = \omega$ and $\mathsf{SVC}^+(S)$ imply that $[S]^\omega$ is not $\sigma$-closed? I.e., is the $\sigma$-closure of $[S]^\omega$ enough to guarantee the countable union theorem?

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