How to tame the Knaster continuum using the ultrafilter orders?

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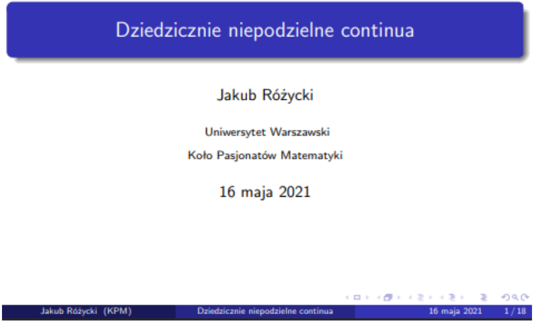
Main information about this topic

- Talk based on my Master's thesis: *Linear orders on chainable continua*, defended in September 2024
- Scientific advisor: prof. Witold Marciszewski
- Idea of the ultrafilter orders: Jakub Różycki



Chainable continua – basic information

Definition

Continuum is a nonempty compact and connected space.

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Definition

Suppose that a topological space X is a continuum. A **chain** in X is a nonempty, finite sequence of nonempty open sets $\mathcal{U} = \{U_1, ..., U_n\}$, satysfying

$$U_i \cap U_j \neq \varnothing \iff |i - j| \leq 1$$

Elements of chain are called **links**.

Definition

Let (X, d) be a metric space and let $\mathcal{A} = \{A_1, ..., A_n\}$ be a family of subsets of X. We define $mesh(\mathcal{A})$ as:

$$mesh(\mathcal{A}) = \max\{diam(A_i) : A_i \in \mathcal{A}\}.$$

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Let (X, d) be a metric continuum. Suppose that $E = \{e_1, ..., e_n\}$ is a chain in continuum X. We say that E is an $\varepsilon-$**chain** in X, if $mesh(E) < \varepsilon$.

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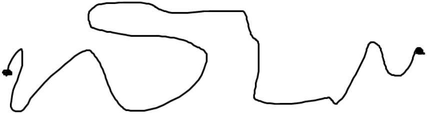
Definition

We say that a metric continuum (X, d) is a **chainable continuum**, if for every $\varepsilon > 0$ there exists ε-chain covering X.

Chainable continua – basic information

Example

An **arc** is any space homeomorphic to the closed interval $[0,1]$



There will be more complicated examples during today's talk.

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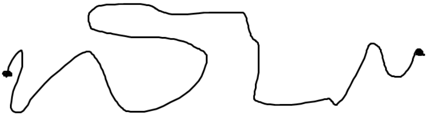

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Not an example (but almost...)

Circle $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

Chainable continua – basic information

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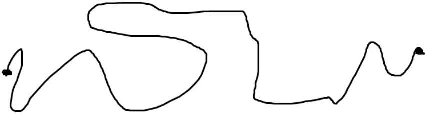

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Not an example (but almost...)

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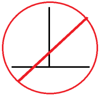
DEFINITELY NON-EXAMPLE

Triod (in the sense of [4]) is any space homeomorphic to letter T.



Observation

Let (X, d) be a metric continuum. Then the following conditions are equivalent.

1. X is a chainable continuum.
2. There exists an infinite sequence of chains $D_1, D_2, D_3, \ldots$ s.t. for every $n \geq 1$ chain D_n covers X and $mesh(D_n) \xrightarrow{n \to \infty} 0$.

Definition

An inverse sequence is a double sequence $(X_i, f_i)_{i=1}^{\infty}$, consisting of topological spaces X_i and continuous functions $f_i : X_{i+1} \to X_i$. An inverse sequence is often presented in the following way:

$$X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} \dots \xleftarrow{f_{i-1}} X_i \xleftarrow{f_i} X_{i+1} \xleftarrow{f_{i+1}} \dots$$

Chainable continua – basic information

Definition

An inverse limit of an inverse sequence $(X_i, f_i)_{i=1}^{\infty}$, denoted by $\varprojlim(X_i, f_i)_{i=1}^{\infty}$, is the subspace of the product space $\prod_{i=1}^{\infty} X_i$, defined by:

$$\varprojlim(X_i, f_i)_{i=1}^{\infty} = \{(x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} X_i : \forall_i \ f_i(x_{i+1}) = x_i\}.$$

For every $n \in \mathbb{N}$ there exists a projection $p_n : \varprojlim(X_i, f_i)_{i=1}^{\infty} \to X_n$ from the inverse limit space to the n-th coordinate of the inverse sequence of topological spaces, given as $p_n((x_i)_{i=1}^{\infty}) = x_n$.

Definition

Let (X, d) be a metric space. We say that $f : X \to [0, 1]$ is an ε-**map**, if f is a continuous surjection and for every $t \in [0, 1]$ $diam(f^{-1}(t)) < \varepsilon$.

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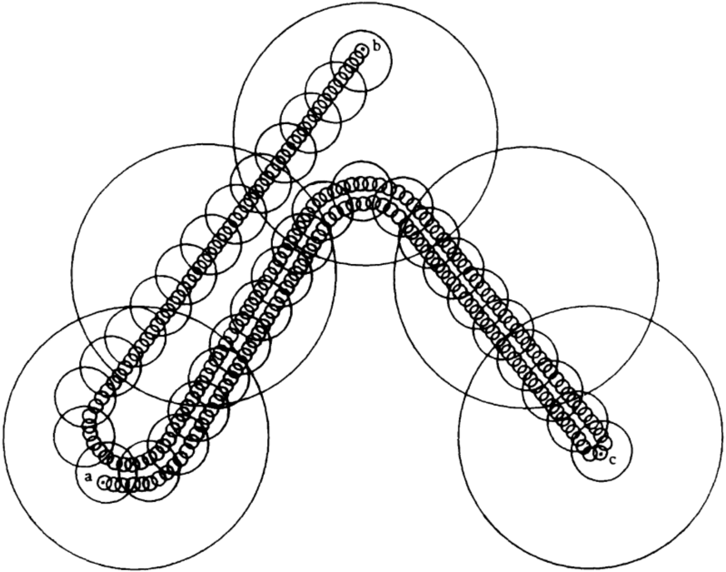
Theorem (proof in e.g. [5],[6])

Suppose that X is a metric continuum. Then the following are equivalent:

- X is chainable,
- for every $\varepsilon > 0$ there is a continuous surjection $f : X \to [0, 1]$, which is an ε-map,
- X is homeomorphic to the inverse limit of arcs which is the space $\varprojlim(X_i, f_i)_{i=1}^{\infty}$ for $X_i = [0, 1]$.

Ultrafilter order on chainable continua

Intuition (image from [6])



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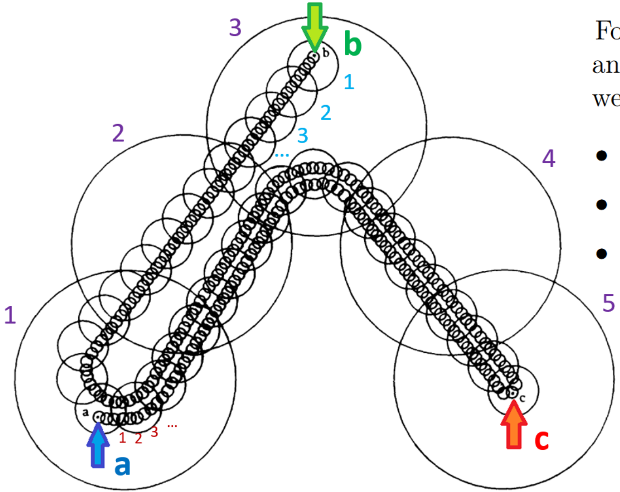

For the points a, b, c and the chains U_1, U_2, U_3 we have:

- $a <_{U_1} b <_{U_1} c$
- $b <_{U_2} a <_{U_2} c$
- $a <_{U_3} c <_{U_3} b$

Notation (see e.g. [1]): If $\{D_n\}_{n=1}^{\infty}$ is a sequence of chains in a space X, then by the symbol

$$d_{i,j}$$

we will denote the $i-th$ link in the $j-th$ chain of the sequence $\{D_n\}_{n=1}^{\infty}$, which is the $i-th$ link of the chain $D_j = \{d_{1,j}, ..., d_{i-1,j}, d_{i,j}, d_{i+1,j}, ..., d_{k_j,j}\}$.

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Definition (the ultrafilter order $\leq_{\mathcal{U}}^{\mathcal{D}}$)

Let X be a chainable continuum and let $\mathcal{D} = \{D_n\}_{n\in\mathbb{N}}$ be a sequence of chains covering X, such that $mesh(D_n) \xrightarrow{n\to\infty} 0$. Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$. Then we can compare any two points $x, y \in X$ in the sense of **the ultrafilter order $\leq_{\mathcal{U}}^{\mathcal{D}}$** on X, which is defined as follows:

$$x \leq_{D_n} y \iff \exists_{\, i\leq j\leq k_n} \; x \in d_{i,n}, \;\; y \in d_{j,n},$$

$$x \leq_{\mathcal{U}}^{\mathcal{D}} y \iff \{n \in \mathbb{N} : x \leq_{D_n} y\} \in \mathcal{U}.$$

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Remark

The above definition depends on the fixed sequence of chains $\{D_n\}_{n\in\mathbb{N}}$ and on fixed nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. So different choices of $\{D_n\}_{n\in\mathbb{N}}$ or $\mathcal{U}$ might generate distinct orders on X.

Ultrafilter order on chainable continua

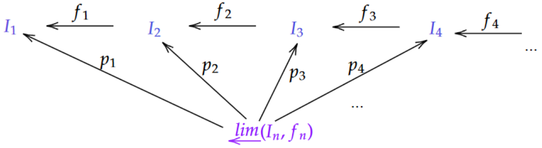
Definition (the ultrafilter order $\leq_{\mathcal{U}}^{\varprojlim(I_i,f_i)_{i=1}^{\infty}}$)

Let $\varprojlim(I_i, f_i)_{i=1}^{\infty}$ be an inverse limit of a sequence of arcs. Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$. Then we can compare any two points $x = (x_i)_{i=1}^{\infty}$, $y = (y_i)_{i=1}^{\infty} \in \varprojlim(I_i, f_i)_{i=1}^{\infty}$ in the sense of **the ultrafilter order** $\leq_{\mathcal{U}}^{(I_i,f_i)_{i=1}^{\infty}}$ on an inverse limit $\varprojlim(I_i, f_i)_{i=1}^{\infty}$, which is defined as follows:

$$x \leq_{I_i} y \iff x_i \leq y_i;$$

$$x \leq_{\mathcal{U}}^{(I_i,f_i)_{i=1}^{\infty}} y \iff \{i \in \mathbb{N} : x \leq_{I_i} y\} \in \mathcal{U},$$

where order $\leq$ is the standard order on the unit closed interval I_i.



Linearity of $\leq_{\mathcal{U}}^{\mathcal{D}}$

For any nonprincipal ultrafilter on $\mathbb{N}$ and for any $\mathcal{D} = \{D_n\}_{n\in\mathbb{N}}$, covering chainable continuum X, such that $mesh(D_n) \xrightarrow{n\to\infty} 0$, order $\leq_{\mathcal{U}}^{\mathcal{D}}$ is a linear order on X.

Linearity of $\leq_{\mathcal{U}}^{(I_i,f_i)_{i=1}^{\infty}}$

Similarly, any ultrafilter order $\leq_{\mathcal{U}}^{(I_i,f_i)_{i=1}^{\infty}}$ on an inverse limit $\varprojlim(I_i, f_i)_{i=1}^{\infty}$ is also a linear order on $\varprojlim(I_i, f_i)_{i=1}^{\infty}$.

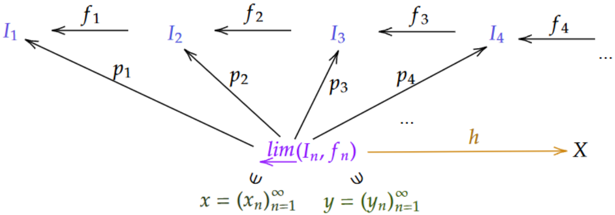
Ultrafilter order on chainable continua

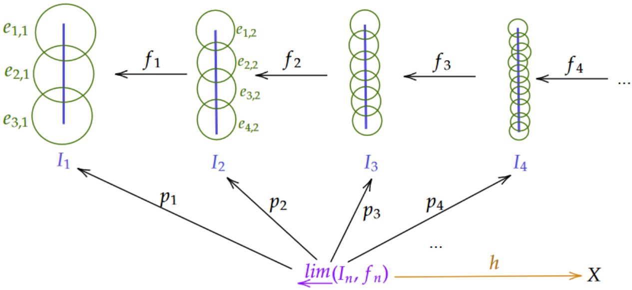
Theorem (W. Marciszewski, J.Ś.)

Let $\varprojlim(I_i, f_i)_{i=1}^{\infty} = Y$ be an inverse limit of a sequence of arcs and let X be a chainable continuum, homeomorphic to Y. Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$. Then any ultrafilter order $\leq_{\mathcal{U}}^{(I_i,f_i)_{i=1}^{\infty}}$ on Y generates an ultrafilter order on X, i.e. there exists an ultrafilter order $\leq_{\mathcal{U}}^{\mathcal{D}}$ on X such that:

$$x \leq_{\mathcal{U}}^{(I_i,f_i)_{i=1}^{\infty}} y \iff h(x) \leq_{\mathcal{U}}^{\mathcal{D}} h(y).$$

Ultrafilter order on chainable continua





Ultrafilter order on chainable continua

Question

Let X be a chainable continuum and let $Y = \varprojlim(I_i, f_i)_{i=1}^{\infty}$ be an inverse limit of a sequence of arcs homeomorphic to X.

Is it true that for any sequence of chains $\mathcal{D} = \{D_n\}_{n\in\mathbb{N}}$, covering X, such that $mesh(D_n) \xrightarrow{n\to\infty} 0$, for any nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and every homeomorphism $h' : X \to Y$ there exists an ultrafilter order $\leq_{\mathcal{U}}^{(I_i,f_i)_{i=1}^{\infty}}$ on Y such that the condition

$$x \leq_{\mathcal{U}}^{\mathcal{D}} y \iff h'(x) \leq_{\mathcal{U}}^{(I_i,f_i)_{i=1}^{\infty}} h'(y)$$

is fulfilled for all $x, y \in X$?

How many distintct orders can be obtained?

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Theorem (W. Marciszewski, J.Ś.)

On any arc there exist exactly 2 distinct ultrafilter orders.

One of them coincides with the natural order on an arc $<$ and another one is opposite to the natural order $<$.

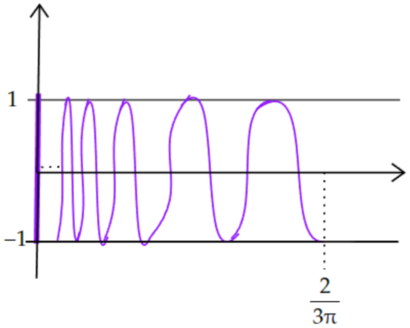
How many distintct orders can be obtained?

Theorem (W. Marciszewski, J.Ś.)

Let S_1 be the Warsaw sine curve, i.e. the chainable continuum defined in the following way: $S_1 = \overline{X}$, where

$$X = \{(x, \sin(\frac{1}{x})) : x \in (0, \frac{2}{3\pi}]\}.$$

There exist exactly 4 distinct ultrafilter orders on S_1.



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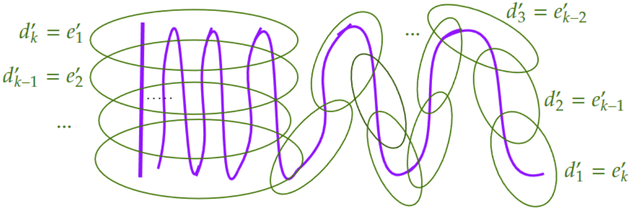
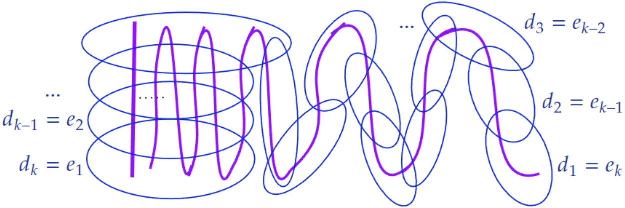
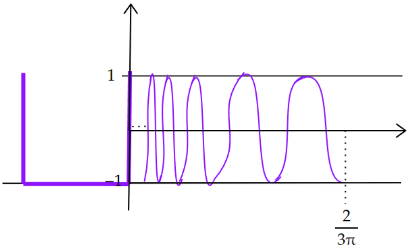




Lemma (really useful!)

Let X be a chainable continuum and let $x, y, z \in X$. Let $\mathcal{D} = \{D_n\}_{n\in\mathbb{N}}$ be any sequence of chains covering X, such that $mesh(D_n) \xrightarrow{n\to\infty} 0$. Suppose that there exists a continuum $M \subseteq X$ such that $x, y \in M$ and $z \notin M$. Then

$$\exists_k \; \forall_{n>k} \neg[(x \leq_{D_n} z \leq_{D_n} y) \vee (y \leq_{D_n} z \leq_{D_n} x)].$$



How many distintct orders can be obtained?

Theorem (W. Marciszewski, J.Ś.)

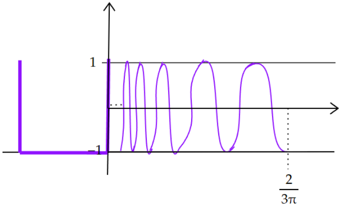
Let S_2 be the modified Warsaw sine curve, i.e. the chainable continuum defined in the following way:

$$S_2 = \overline{X} \cup (\{-1\} \times [-1,1]) \cup ([-1,0] \times \{-1\}),$$

where

$$X = \{(x, \sin(\frac{1}{x})) : x \in (0, \frac{2}{3\pi}]\}.$$

There exist exactly 2 distinct ultrafilter orders on S_2.



How many distintct orders can be obtained?

Definition

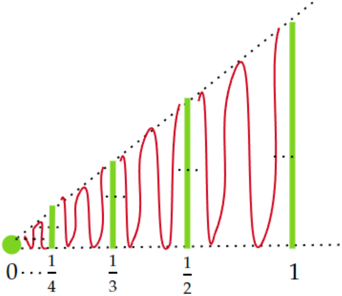
Consider another chainable continuum, called S_3, which we define as follows:

$$S_3 = \bigcup_{n \in \mathbb{N} \setminus \{0\}} I_n \cup \bigcup_{n \in \mathbb{N} \setminus \{0\}} A_n \cup \{(0,0)\},$$

where:

- I_n is a closed interval of the form: $\{\frac{1}{n}\} \times [0, \frac{1}{n}]$,
- $A_n = \{\left|x \cdot sin\left(\frac{1}{(x - \frac{1}{n+1})(\frac{1}{n} - x)}\right)\right| : x \in (\frac{1}{n+1}, \frac{1}{n})\}$.

How many distintct orders can be obtained?



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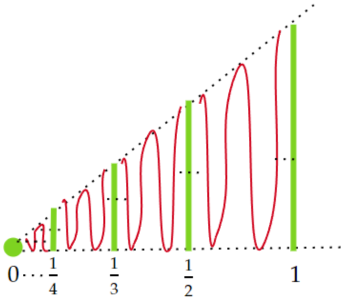
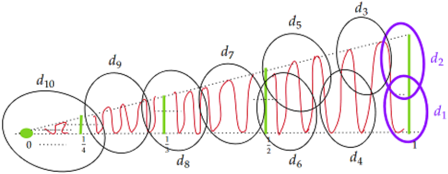


Theorem (W. Marciszewski, J.Ś.)

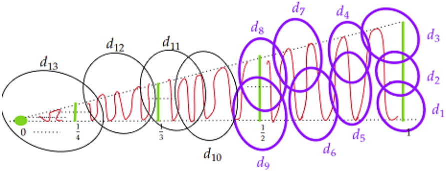
There exist exactly $\mathfrak{c}$ distinct ultrafilter orders on S_3.

How many distinct orders can be obtained?

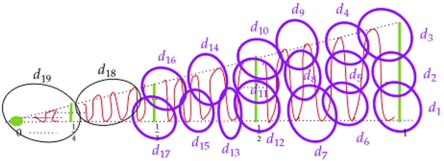
Example: Let $x \in \{0,1\}^{\mathbb{N}}$, $x = 0, 1, 1, \ldots$



0



0, 1



0,1,1

The Knaster continuum

Definition [3]

Let $\mathcal{C} \subseteq [0,1] \times \{0\} \subseteq \mathbb{R}$ be the standard Cantor set.
The Knaster continuum is defined as a subspace of $\mathbb{R}^2$, consisting of:

- all semi-circles with ordinates ≥ 0, with center $(\frac{1}{2}, 0)$ and passing through every point of the Cantor set $\mathcal{C}$,
- all semi-circles with ordinates ≤ 0, which have for $n \geq 1$ the center at $(\frac{5}{2 \cdot 3^n}, 0)$ and pass through each point of the Cantor set $\mathcal{C}$, lying in the interval $[\frac{2}{3^n}, \frac{1}{3^{n-1}}]$.

The Knaster continuum

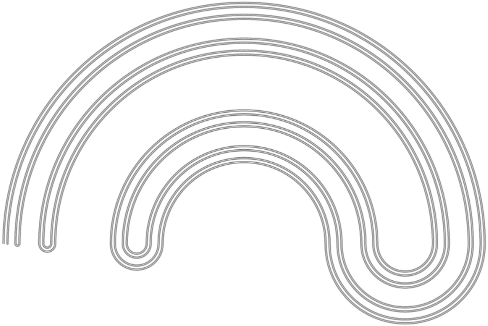
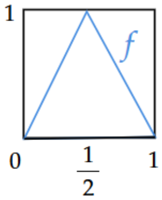


image from https://commons.wikimedia.org/wiki/File:The_Knaster_bucket-handle_continuum.svg

Definition [6]

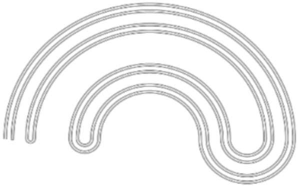
In the book [6] the Knaster continuum was defined in an alternative way – it is a space homeomorphic to the inverse limit of the sequence of arcs $\varprojlim([0,1], f_i)_{i=1}^{\infty}$, where for each i, $f_i = f$ and the map $f : [0,1] \to [0,1]$ is given as:

$$f(t) = \begin{cases} 2t & \text{for } t \in [0, \frac{1}{2}], \\ -2t+2 & \text{for } t \in [\frac{1}{2}, 1]. \end{cases} \quad (1)$$

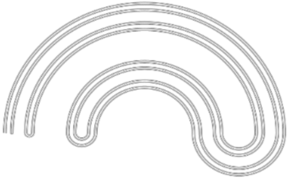


The Knaster continuum

$$\varprojlim \left([0,1] \longleftarrow [0,1] \longleftarrow [0,1] \longleftarrow \dots \right) \overset{homeo}{\simeq}$$



$$\varprojlim \left([0,1] \xleftarrow{\triangle} [0,1] \xleftarrow{\triangle} [0,1] \xleftarrow{\triangle} \dots \right) \overset{homeo}{\simeq}$$



Definition/ interesting property

We say that a continuum X is **indecomposable** if it is not the union of its two proper subcontinua.
Knaster continuum is an example of an indecomposable continuum.

Theorem (W. Marciszewski, J.Ś.)

There exist exactly $2^{\mathfrak{c}}$ distinct ultrafilter orders on the Knaster continuum.

Definition

Let X be a continuum and let $x \in X$. We say that a **composant** of a point x in X is the union of all proper subcontinua containing x.

- Every composant of a continuum X is dense in X.

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- Every composant of a continuum X is dense in X.

Some facts about composants in the Knaster continuum [3]

- There are $\mathfrak{c}$ composants in the Knaster continuum K and they coincide with arc components of K.

Some facts about composants in the Knaster continuum [3]

- The composant of a point $(0, 0)$ in the Knaster continuum is a continous and one-to-one image of the half-line.
- The other composants of the Knaster continuum are continous and one-to-one images of the open interval.

Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$. Let $\mathcal{D} = \{D_n\}_{n\in\mathbb{N}}$ (where $D_n = \{d_{i,n}\}_{i=1}^{k_n}$) be such a sequence of chains covering the Knaster continuum, that for every n, $(0,0) \in d_{1,n}$ and $mesh(D_n) \xrightarrow{n\to\infty} 0$.

Let $\tau_{\mathcal{U}}^{\mathcal{D}}$ be the order topology generated by the ultrafilter order $\leq_{\mathcal{U}}^{\mathcal{D}}$.

Some facts about composants in K (continued)

Notice that

- The composant of the Knaster continuum containing point $(0,0)$ in the space $(K, \tau_{\mathcal{U}}^{\mathcal{D}})$ has the order type of the interval $[0,1)$ (and the point $(0,0)$ is the smallest in the sense of order $\leq_{\mathcal{U}}^{\mathcal{D}}$).

Some facts about composants in K (continued)

Notice that

- The composant of the Knaster continuum containing point $(0,0)$ in the space $(K, \tau_{\mathcal{U}}^{\mathcal{D}})$ has the order type of the interval $[0,1)$ (and the point $(0,0)$ is the smallest in the sense of order $\leq_{\mathcal{U}}^{\mathcal{D}}$).
- All the other composants in $(K, \tau_{\mathcal{U}}^{\mathcal{D}})$ have the order type of an open interval.

Some facts about composants in K (continued)

Notice that

- The composant of the Knaster continuum containing point $(0,0)$ in the space $(K, \tau_{\mathcal{U}}^{\mathcal{D}})$ has the order type of the interval $[0,1)$ (and the point $(0,0)$ is the smallest in the sense of order $\leq_{\mathcal{U}}^{\mathcal{D}}$).
- All the other composants in $(K, \tau_{\mathcal{U}}^{\mathcal{D}})$ have the order type of an open interval.
- All composants of the Knaster continuum are open in $(K, \tau_{\mathcal{U}}^{\mathcal{D}})$.

Some facts about composants in K (continued)

Notice that

- The composant of a Knaster continuum containing point $(0,0)$ in the space $(K, \tau_{\mathcal{U}}^{\mathcal{D}})$ has the order type of the interval $[0,1)$ (and the point $(0,0)$ is the smallest in the sense of order $\leq_{\mathcal{U}}^{\mathcal{D}}$).
- All the other composants in $(K, \tau_{\mathcal{U}}^{\mathcal{D}})$ have the order type of an open interval.
- All composants of the Knaster continuum are open in $(K, \tau_{\mathcal{U}}^{\mathcal{D}})$.

The following theorem follows from the above observations.

The Knaster continuum

Theorem (W. Marciszewski, J.Ś.)

Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$. Let $\mathcal{D} = \{D_n\}_{n\in\mathbb{N}}$ (where $D_n = \{d_{i,n}\}_{i=1}^{k_n}$) be such a sequence of chains covering the Knaster continuum, that for every n, $(0,0) \in d_{1,n}$ and $mesh(D_n) \xrightarrow{n\to\infty} 0$.

Then the Knaster continuum with the order topology $\tau_{\mathcal{U}}^{\mathcal{D}}$, generated from an ultrafilter order $\leq_{\mathcal{U}}^{\mathcal{D}}$, is homeomorphic to the disjoint sum of the topological spaces X_i:

$$(K, \tau_{\mathcal{U}}^{\mathcal{D}}) \overset{\text{homeo}}{\simeq} \bigoplus_{i\in I} X_i,$$

where X_0 is a space homeomomorphic to the interval $[0,1)$, corresponding to the arc component of the Knaster continuum containing the point $(0,0)$, and all other X_i are homeomorphic to the open interval $(0,1)$ and correspond to the remaining arc components of the Knaster continuum.

Corollary

The Knaster continuum endowed with the order topology $\tau_{\mathcal{U}}^{\mathcal{D}}$, is a metrizable, non-connected, non-compact and non-separable space.

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Thank you!


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miotełka
nieskoń-czone kieszenie
pasek do regulacji mocy nieskończonych kieszeni
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