Partition relations and coloured finite digraphs

03E02, 05C15, 05C20

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Definition

$\alpha \to (\beta, n)$ means $\forall c : [\alpha]^2 (\exists X \in [\alpha]^\beta : c``([X]^2) = 0 \vee \exists X \in [\alpha]^n : c``([X]^2) = 1$.

Remark

Here we are always referring to the order-type, i.e. $[\gamma]^\delta$ *is the set of all subsets of* γ *whose order-type is* δ*.*

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Fact

For any linear order φ *we have both* $\varphi \not\to (\omega + 1, \omega)$ *and* $\varphi \not\to (\omega^*, \omega)$*.*

Remark

Both statements can be simultaneously proved by enumerating φ *in ordertype* ω *and colouring a pair in colour 0 iff both orders agree on it.*

Theorem (Erdős, Rado(slightly different formulation))

The partition relation $\omega l \to (\omega m, n)$—with $l, m, n < \omega$—holds true if and only if every directed graph $D = \langle l, A\rangle$ contains an independent set of size m or there is a complete subdigraph of D induced by a set of n vertices without a cycle.

Proof-idea

$$\begin{aligned} \chi' : [\omega]^2 &\longrightarrow 2^{l^2} \\ \{j,k\}_< &\longmapsto \sum_{h,i<l} \chi(\{\omega h + j, \omega i + k\}) \cdot 2^{h \cdot l + i} \end{aligned}$$

Note that...

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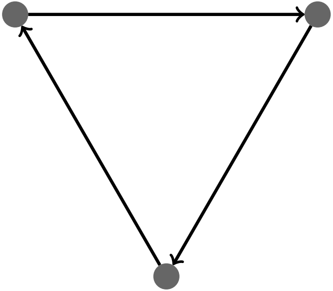
Note that...

- *...we can assume without loss of generality that all ω-blocks are homogeneous of colour 0.*
- *...we may also assume wlog that there is only one arc between each two points of the digraph.*

Example

Let α be the least ordinal such that $\alpha \to (\omega 2, 3)$. Then $\alpha = \omega 4$.



Theorem (Specker)

$\omega^2 \to (\omega^2, n)$ for all natural n.

Definition

Let $\langle v_0, \ldots, v_n \rangle$ be a closed walk in the symmetrization of a coloured digraph $\langle V, A, c \rangle$ where $\mathrm{ran}(c) = 3$. We may first assume that $\langle v_0, v_1 \rangle \in A$. If not we may look instead at the walk $\langle v_1, v_n, v_{n-1}, \ldots, v_1 \rangle$. The original walk will be called agreeable if and only if the second one will.
We will follow the walk, at each step $i \leqslant n$ associating a state $s(i) < 3$. The walk is *agreeable* if the state at the end of the walk, i.e. $s(n)$ will be higher than at its beginning—i.e. $s(0)$—in both cases $s(0) = 0$ and $s(0) = 1$. The state changes according to the following rules:

- If $s(i) = 2$ then $s(i+1) = 2$.
- If $s(i) = 1$ and
  - $\langle v_i, v_{i+1} \rangle \in A$ then $s(i+1) = c(\langle v_i, v_{i+1} \rangle)$
  - $\langle v_{i+1}, v_i \rangle \in A$ then $s(i+1) = \min\{c(\langle v_{i+1}, v_i \rangle) + 1, 2\}$.
- If $s(i) = 0$ and
  - $\langle v_i, v_{i+1} \rangle \in A$ then $s(i+1) = 0$.
  - $\langle v_{i+1}, v_i \rangle \in A$ then $s(i+1) = \max\{c(\langle v_{i+1}, v_i \rangle), 1\}$.

When a closed walk is not agreeable we call it *disagreeable*.

Theorem (W.?, 2011)

The partition relation $\omega^2 l \to (\omega^2 m, n)$ holds true if and only if every coloured digraph $C = \langle l, A, c\rangle$ with $\mathrm{ran}(c) = 3$ contains an independent set of size m or there is a complete subdigraph of C induced by a set of n vertices such that all closed walks in C's symmetrization are agreeable.

Proof-idea

$$\begin{aligned}\chi' : [\omega]^4 &\longrightarrow 2^{3l^2}\\ \{h,i,j,k\}_< &\longmapsto \sum_{f,g<l} \Big(\chi(\{\omega^2 f + \omega h + i, \omega^2 g + \omega j + k\})2^{3(lf+g)}\\ &\qquad + \chi(\{\omega^2 f + \omega h + j, \omega^2 g + \omega i + k\})2^{3(lf+g)+1}\\ &\qquad + \chi(\{\omega^2 f + \omega h + k, \omega^2 g + \omega i + j\})2^{3(lf+g)+2}\Big)\end{aligned}$$

Without loss of generality we may assume the following:

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Without loss of generality we may assume the following:

- *For all* $\omega y + z < \omega^2 l$ *with* $y, z < \omega$ *we have* $y < z$.

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- *All ω-blocks and in fact...*

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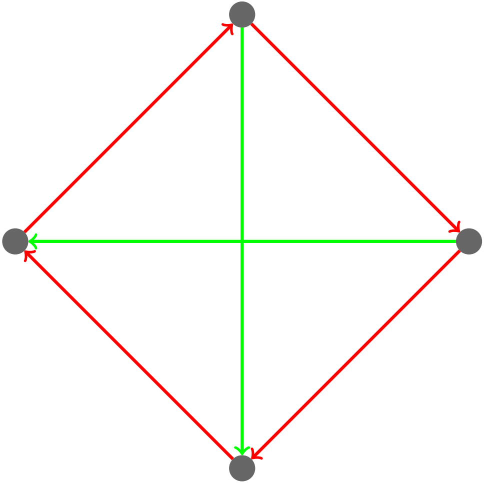
Without loss of generality we may assume the following:

- *For all $\omega y + z < \omega^2 l$ with $y, z < \omega$ we have $y < z$.*
- *All ω-blocks and in fact... all ω^2-blocks are homogeneous of colour 0.*
- *There is only one arc between any two points in the graph.*

Example

Let $\alpha < \omega^2 5$. Then $\alpha \not\to (\omega^2 2, 3)$.

Now identify 0 with red, 1 with yellow and 2 with green!



Theorem (Specker)

$\omega^m \to (\omega^m, 3)$ *implies* $m \notin \omega \setminus 3$.

Theorem (Milner)

$\omega^\omega \to (\omega^\omega, n)$ *for all natural* n.

Theorem (Darby, Larson)

$\omega^{\omega^2} \to (\omega^{\omega^2}, 4)$ *but*

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$\omega^{\omega^2} \to (\omega^{\omega^2}, 4)$ but $\omega^{\omega^2} \not\to (\omega^{\omega^2}, 5)$.

Question (Handbook)

Does $\omega^{\omega^3} \to (\omega^{\omega^3}, 3)$?

Question

Does $\omega^{\omega^{\omega^\omega}} \to (\omega^{\omega^{\omega^\omega}}, n)$ for all natural n?

Question

Does $\omega_1^{CK} \to (\alpha, 3)$ *for every recursive* α*?*

Conjecture

Similar to the construction before and inspired by the proof of $\omega^\omega \to (\omega^\omega, n)$ *for all natural* n *one can find a recursive characterization of* $\omega^{\omega l} \to (\omega^{\omega m}, n)$ *for natural* l*,* m *and* n*.*

Thank you for your attention!