Kurepa and weak Kurepa trees

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Recall the following definitions:

Definition

We say that an ω_1-tree T is *Aronszajn* if it has no cofinal branches.

Definition

We say that an ω_1-tree T is *Suslin* if it is Aronszajn and does not contain uncountable antichains.

Definition

We say that an ω_1-tree T is *Kurepa* if it has at least ω_2-many cofinal branches.

Kurepa hypothesis

We say that the *Kurepa Hypothesis*, KH, holds if there exists a Kurepa tree.

- (Jensen) $\Diamond^+$ implies the existence of a Kurepa tree.
- (Silver) Silver proved the consistency of $\neg$KH by showing that after forcing with the Lévy collapse to turn an inaccessible cardinal κ into ω_2, there are no Kurepa trees.
- (Solovay) The inaccessible cardinal is necessary because if ω_2 is not an inaccessible cardinal in L then there exists a Kurepa tree.

Adding a Kurepa tree

- Jin and Shelah asked whether it is consistent that CH holds, there are no Kurepa trees, and there exists a *ccc* forcing of *size at most* ω_1 which adds a Kurepa tree.
- This question was motivated in part by Shelah's result that adding a single Cohen real adds a Suslin tree.
- The model is not a Lévy collapse of an inaccessible cardinal which becomes the new ω_2.

A result of Jensen and Schlechta

- (Jensen) If $\square_{\omega_1}$ holds then there is a ccc forcing which adds an ω_1-Kurepa tree. However this ccc forcing has size ω_2.
- (Jensen and Schlechta) There is a model where $\neg$KH holds and it is indestructible under all ccc forcings.
- The model is the Lévy collapse of a Mahlo cardinal which becomes the new ω_2.
- Note that $\neg$KH holds already in the Lévy collapse of an inaccessible cardinal but for the indestructibility under ccc forcings a Mahlo cardinal is needed.

Result of Jin and Shelah

- Jin and Shelah proved that assuming the existence of an inaccessible cardinal κ, there exists a forcing which preserves ω_1, makes κ the new ω_2, forces that there are no Kurepa trees, and introduces a countably distributive Aronszajn tree which as a forcing notion produces a Kurepa tree.

Almost Kurepa Suslin tree

In the Jin-Shelah model, forcing with the Aronszajn tree introduces another tree which is a Kurepa tree.

Now we define a stronger concept:

Definition

We say that an ω_1-tree T is an *almost Kurepa Suslin tree* if it is a Suslin tree which becomes a Kurepa tree after forcing with it.

Note: When forcing with a tree T, the order is the reverse order of the tree T.

Existence of an almost Kurepa Suslin tree

- (Jensen) $\Diamond^+$ implies that there is an almost Kurepa Suslin tree.
- (Bilaniuk) $\Diamond$ and KH imply that there is an almost Kurepa Suslin tree.
- Bilaniuk constructed an almost Kurepa Suslin tree S with at least ω_2-many automorphisms indexed by cofinal branches of a fixed Kurepa tree T: $\{\pi_b \,|\, b \text{ is a cofinal branch } T\}$. The automorphisms are non-trivial in the sense that for all $b \neq b'$ and $s \in S$ there is $t \geq s$ such that $\pi_b(t) \neq \pi_{b'}(t)$.
- Bilaniuk asked whether it is consistent that $\Diamond$ holds, there are no Kurepa trees, and there exists a Suslin tree with at least ω_2-many automorphisms.
- Related to this question is the question whether it is consistent that there exists an almost Kurepa Suslin tree and there are no Kurepa trees.

Lévy collapse

In generic extensions by the Lévy collapse which turns an inaccessible cardinal κ into ω_2 the following hold:

- (Silver) There are no Kurepa trees.
- (Folklore) There is no ccc forcing of size at most ω_1 which adds an Kurepa tree; consequently there are no almost Kurepa Suslin trees.
- (Bilaniuk) Every Suslin tree has at most ω_1-many automorphisms.

Our result

Theorem (Krueger, S.)

Suppose that there exists an inaccessible cardinal κ and there exists an infinitely splitting normal free Suslin tree S. Then there exists a forcing poset $\mathbb{P}$ such that the product $\mathrm{Coll}(\omega_1, <\kappa) \times \mathbb{P}$ forces:

- $\kappa = \omega_2$;
- GCH;
- *S is Suslin;*
- *there exist an almost disjoint family $\{f_\alpha \mid \alpha < \omega_2\}$ of automorphisms of S;*
- *there are no Kurepa trees.*

Corollaries of our result:

1. It is consistent that CH holds, there are no Kurepa trees, and there exists a ccc forcing of size at most ω_1 which adds a Kurepa tree.
2. It is consistent that there exists an almost Kurepa Suslin tree and there are no Kurepa trees.
3. It is consistent that $\Diamond$ holds, there are no Kurepa trees, and there exists a Suslin tree with at least ω_2-many automorphisms.
4. It is consistent that there exists a non-saturated Aronszajn tree and there are no Kurepa trees.

Cohen forcing and Kurepa trees

To our knowledge it is open whether $\neg$KH is preserved by a single Cohen forcing.

Weak Kurepa trees

Recall the following definition:

Definition

We say that a tree T with size and height ω_1 is a *weak Kurepa tree* if it has at least ω_2-many cofinal branches. We say that the *weak Kurepa Hypothesis*, wKH, holds if there exists a weak Kurepa tree on ω_1.

The negation of the weak Kurepa hypothesis

Some basic properties:

- If CH holds, then $2^{<\omega_1}$ is a weak Kurepa tree.
- Therefore $\neg$wKH implies $2^\omega > \omega_1$.
- (Mitchell) In the generic extension by Mitchell forcing up to an inaccessible cardinal $\neg$wKH holds.
- (Silver) The inaccessible cardinal is necessary. If $\neg$wKH holds, then ω_2 is inaccessible in L.
- (Baumgartner) If $\neg$wKH holds, then $2^\omega = \omega_2$ implies $2^{\omega_1} = \omega_2$; in fact, even $\diamondsuit^+(\omega_2 \cap \mathrm{cof}(\omega_1))$ holds.
- Baumgartner's result can be generalized as follows: if $2^\omega < \aleph_{\omega_1}$, then $2^{\omega_1} = 2^\omega$.

In our work with Radek Honzik and Chris Lambie Hanson we showed that the negation of weak Kurepa hypothesis is preserved by all σ-centered forcings.

Theorem (Honzik, Lambie-Hanson, S.)

GMP *implies* $\neg$wKH *is preserved by all σ-centered forcings, i.e. if V is a transitive model satisfying* GMP, $\mathbb{P} \in V$ *is σ-centered, and G is $\mathbb{P}$-generic over V, then $V[G]$ satisfies* $\neg$wKH. *In particular,* $\neg$wKH *is preserved over models of* GMP *by adding any number of Cohen subsets of ω.*

In fact, the argument needs only a guessing model property for small sets which turns out to be equivalent to $\neg$wKH: $\neg$wKH is therefore always preserved by σ-centered forcings.

$\neg$KH vs $\neg$wKH

- $\neg$wKH is indestructible by all σ-centered forcings, in particular it is indestructible by Cohen forcing of any length (if a weak Kurepa tree is added, it is added by a small part of Cohen which is σ-centered). On the other hand, it is open whether $\neg$KH itself is indestructible by one Cohen forcing.
- Note that $\neg$wKH implies $\neg$KH, hence if there is a model where $\neg$KH is destructible by adding one Cohen subset of ω, then there has to be a weak Kurepa tree in that model.
- (Jensen and Schlechta) Under the assumption of Mahlo cardinal, there is a model (Lévy collapse) where $\neg$KH holds and it is indestructible under all ccc forcings. The analogous question is open for $\neg$wKH, with the Mitchell model instead of the Lévy model. We were able to get a partial result in this direction.

Mitchell forcing

The Mitchell forcing is the standard way of collapsing a large cardinal κ to a double succesor of a regular cardinal μ while adding κ-many new subsets of μ. For this lecture μ will be just ω.

Theorem (Honzik, S.)

Suppose κ is Mahlo. Then $\neg$wKH *is indestructible in* $V[\mathbb{M}(\omega,\kappa)]$ *under all ccc forcings which live in* $V[\mathrm{Add}(\omega,\kappa)]$.

Note that $V[\mathrm{Add}(\omega,\kappa)] \subseteq V[\mathbb{M}(\omega,\kappa)]$ so the statement of the theorem makes sense.

The proof is based on an indestructibility result of Jensen and Schlechta. We need to assume that the ccc forcing lives in $V[\mathrm{Add}(\omega,\kappa)]$ for the quotient analysis.

A result of Todorcevic

Suppose κ is inaccessible. Todorcevic showed that $\neg$wKH is indestructible over the Mitchell model $V[\mathbb{M}(\omega, \kappa)]$ under any finite-support iteration of length κ of ccc forcing notions which have size at most ω_1 and do not add new cofinal branches to ω_1-trees.

The proof uses the fact that small forcings of size ω_1 are added by initial segments of the Mitchell iteration, so a simplified quotient analysis can be done.

1. Is $\neg$KH indestructible by one Cohen forcing?
   - Note that $\neg$wKH is indestructible by Cohen forcing.

2. Is there a model where $\neg$wKH holds and all ccc forcings preserve it? In particular, is $\neg$wKH consistent with MA_{ω_2}?
   - Note that $\neg$wKH is consistent with MA_{ω_1} (Todorcevic).
   - Moreovere note that there is a model where $\neg$KH is indestructible by all ccc forcings and in particular $\neg$KH is consistent with MA_{ω_2} (Jensen and Schlechta).