# Rearrangement & Subseries numbers

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*Winter School 2025*
Hejnice, Czech Republic

January 27, 2025

Given a sequence $\boldsymbol{a} = \langle a_n \mid n \in \omega \rangle \in {}^{\omega}\mathbf{R}$ with $\lim_{n\to\infty} a_n = 0$, we may consider the infinite series $\sum \boldsymbol{a} = \sum_{n\in\omega} a_n$ and the sequence of partial sums $\langle P_n \mid n \in \omega \rangle$, where $P_n = \sum_{k\leq n} a_n$.

It is possible that $\lim_{n\to\infty} P_n$ is equal to a real number, called its limit, in which case we say $\sum \boldsymbol{a}$ *converges*. If $\sum_{n\in\omega} |a_n|$ converges, then $\sum \boldsymbol{a}$ is *absolutely convergent*, or else, $\sum \boldsymbol{a}$ is *conditionally convergent*.

Otherwise $\sum \boldsymbol{a}$ *diverges*. Either $\lim_{n\to\infty} P_n$ equals $\infty$ or $-\infty$, in which case $\sum \boldsymbol{a}$ *tends to infinity*, or $\langle P_n \mid n \in \omega \rangle$ has multiple accumulation points, in which case $\sum \boldsymbol{a}$ *oscillates*.

Let $\mathcal{S}_\omega$ be the set of permutations (= bijections) $\pi : \omega \to \omega$, and for $\pi \in \mathcal{S}_\omega$ we will write $\sum \boldsymbol{a}_\pi = \sum_{n\in\omega} a_{\pi(n)}$.

**Theorem** *Riemann*

If $\sum \boldsymbol{a}$ is conditionally convergent and $r \in \mathbf{R}$, then

- there is $\pi \in \mathcal{S}_\omega$ s.t. $\sum \boldsymbol{a}_\pi = r$,
- there is $\rho \in \mathcal{S}_\omega$ s.t. $\sum \boldsymbol{a}_\rho$ tends to ($\pm$) infinity,
- there is $\sigma \in \mathcal{S}_\omega$ s.t. $\sum \boldsymbol{a}_\sigma$ diverges by oscillation

In 2015, Michael Hardy asked on *MathOverflow*:

How large does a subset $C \subseteq \mathcal{S}_\omega$ have to be, such that for every conditionally convergent $\sum \boldsymbol{a}$ there exists some $\pi \in C$ for which $\sum \boldsymbol{a}_\pi$ does not converge to the same value as $\sum \boldsymbol{a}$?

The answer turns out to be quite interesting, and became subject of a paper by Blass, Brendle, Brian, Hamkins, Hardy, and Larson (2019).

To spoil the answer to Hardy's original question:

$$\max\{\mathfrak{b}, \operatorname{cov}(\mathcal{N})\} \leq \mathfrak{rr} \leq \operatorname{non}(\mathcal{M}),$$

where the *rearrangement number* $\mathfrak{rr}$ is the least cardinality of such $C$.

Let $[\omega]^\omega$ be the set of infinite subsets of $\omega$, and for $X \in [\omega]^\omega$ we will write $\sum_X \boldsymbol{a} = \sum_{n \in X} a_n$ (ordered in the natural order $X$ inherits from $\omega$).

**Theorem**

If $\sum \boldsymbol{a}$ is conditionally convergent and $r \in \mathbf{R}$, then

- there is $X \in [\omega]^\omega$ s.t. $\sum_X \boldsymbol{a} = r$,
- there is $Y \in [\omega]^\omega$ s.t. $\sum_Y \boldsymbol{a}$ tends to ($\pm$) infinity,
- there is $Z \in [\omega]^\omega$ s.t. $\sum_Z \boldsymbol{a}$ diverges by oscillation

Two months after Hardy's question, Rahman Mohammadpour asked on *MathOverflow* whether weakening "permutations" to "injections" in the definition of $\mathfrak{rr}$ results in a consistently different cardinal characteristic.

The answer became subject to another paper by Brendle, Brian, and Hamkins (2019).

Let $\mathfrak{ß}$ be the least size of a family $D \subseteq [\omega]^\omega$ such that for every conditionally convergent $\sum \boldsymbol{a}$ there exists some $X \in D$ such that $\sum_X \boldsymbol{a}$ diverges. We have

$$\max\{\mathfrak{s}, \mathrm{cov}(\mathcal{N})\} \leq \mathfrak{ß} \leq \mathrm{non}(\mathcal{M}).$$

The answer to Mohammadpour's original question is the *subrearrangement number* $\mathfrak{sr}$, and $\mathfrak{sr} = \min\{\mathfrak{ß}, \mathfrak{rr}\}$.

In the original paper, the authors used the symbol ß for the subseries number. This letter, the *Eszett* of the German language, commonly replaces "ss".

But cardinal characteristics are usually displayed using fraktur script! So what do we do?

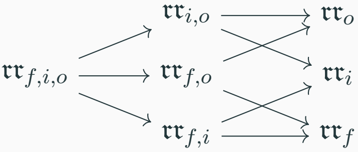
1. Import the package `yfonts`,
2. Now we have a fraktur medial s at our disposal: ſ
3. Glue it together with the fraktur $\mathfrak{z}$
4. Subseries number: $\mathfrak{ſz}$.

We will define the following *convergence behaviours*:

- $f$ “converges to a different limit”
- $i$ “tends to infinity”
- $o$ “oscillates”

For $\Gamma$ a set of convergence behaviours, we can define a rearrangement number $\mathfrak{rr}_\Gamma$ as the least cardinality of a set $C \subseteq \mathcal{S}_\omega$ such that for every CC $\sum \boldsymbol{a}$ there exists $\pi \in C$ such that $\sum \boldsymbol{a}_\pi$ behaves according to one of the elements of $\Gamma$.

For instance, the original $\mathfrak{rr}$ is equal to $\mathfrak{rr}_{f,i,o}$. We observe:




(An arrow $\mathfrak{x} \to \mathfrak{y}$ stands for $\mathfrak{x} \leq \mathfrak{y}$)

# Are the rearrangement numbers all different?

No.

Let $\pi \in \mathcal{S}_\omega$, then we call $\tau_\pi \in \mathcal{S}_\omega$ a *mixing* of $\pi$ with the identity if $\tau_\pi[n] = \pi[n]$ for infinitely many $n$, and $\tau_\pi[k] = k$ for infinitely many $k$.

**Theorem** *Blass, Brendle, Brian, Hamkins, Hardy, and Larson 2019*

$\mathfrak{rr}_{f,i,o} = \mathfrak{rr}_{f,o} = \mathfrak{rr}_{i,o} = \mathfrak{rr}_o$.

*Proof.* It suffices to show $\mathfrak{rr}_o \leq \mathfrak{rr}_{f,i,o}$.

Let $C \subseteq \mathcal{S}_\omega$ witness $\mathfrak{rr}_{f,i,o}$ and $\sum \boldsymbol{a} = r$ be CC. If there is $\pi \in C$ such that $\sum \boldsymbol{a}_\pi = r' \in \mathbf{R} \cup \{\infty, -\infty\}$ and $r' \neq r$, then $\sum \boldsymbol{a}_{\tau_\pi}$ has both $r$ and $r'$ as accumulation points, and thus oscillates.

Therefore $C \cup C'$, where $C' = \{\tau_\pi \mid \pi \in \mathcal{S}_\omega\}$, witnesses $\mathfrak{rr}_o$. $\square$

A *relational system* is a triple $\mathscr{R} = \langle R, X, Y\rangle$ where $R \subseteq X \times Y$. We call $X$ the set of *challenges* and $Y$ the set of *responses*. A response $y$ *meets* the challenge $x$ if $x \ R \ y$.

We define two cardinal characteristics:

$$\mathfrak{D}(R, X, Y) = \min\{|D| \mid D \subseteq Y \text{ and } \forall x \in X \exists y \in D(x \ R \ y)\}$$
$$\mathfrak{B}(R, X, Y) = \min\left\{|B| \mid B \subseteq X \text{ and } \forall y \in Y \exists x \in B(x \not R \ y)\right\}$$

Note that $\mathfrak{B}(R, X, Y) = \mathfrak{D}(\not R^{-1}, Y, X)$. Thus, $\mathscr{R}^\perp = \langle \not R^{-1}, Y, X\rangle$ is called the *dual* relational system of $\mathscr{R} = \langle R, X, Y\rangle$.

If $\mathscr{R} = \langle R, X, Y\rangle$ and $\mathscr{S} = \langle S, A, B\rangle$ are relational systems, then a *Tukey connection* is a pair of maps $\rho_- : X \to A$ and $\rho_+ : B \to Y$ such that for any $x \in X$ and $b \in B$ for which $\rho_-(x)\ S\ b$ holds, also $x\ R\ \rho_+(b)$ holds. If a Tukey connection from $\mathscr{R}$ to $\mathscr{S}$ exists, we write this as $\mathscr{R} \preceq \mathscr{S}$.

**Lemma**

$$\mathscr{R} \preceq \mathscr{S} \text{ implies } \begin{cases} \mathfrak{D}(R, X, Y) \le \mathfrak{D}(S, A, B), \quad \text{and} \\ \mathfrak{B}(R, X, Y) \ge \mathfrak{B}(S, A, B). \end{cases}$$

Let us define some more convergence behaviours:

- $c$ “converges”
- $ac$ “converges absolutely”
- $cc$ “converges conditionally”

For $\Gamma$ a set of convergence behaviours, define $\mathfrak{S}_\Gamma$ as the set of $\boldsymbol{a} \in {}^\omega\mathbf{R}$ such that $\sum \boldsymbol{a}$ behaves according to an element of $\Gamma$.

Let $R_\Gamma \subseteq \mathfrak{S}_{cc} \times \mathcal{S}_\omega$ be the relation defined by $\boldsymbol{a}\ R_\Gamma\ \pi$ if and only if $\boldsymbol{a}_\pi \in \mathfrak{S}_\Gamma$. Now note that $\mathfrak{rr}_\Gamma = \mathfrak{D}(R_\Gamma, \mathfrak{S}_{cc}, \mathcal{S}_\omega)$. We will write $\mathscr{R}_\Gamma = \langle R_\Gamma, \mathfrak{S}_{cc}, \mathcal{S}_\omega \rangle$.

Let $\mathfrak{rr}_\Gamma^\perp = \mathfrak{B}(R_\Gamma, \mathfrak{S}_{cc}, \mathcal{S}_\omega)$, which is the least size of a set $A \subseteq \mathfrak{S}_{cc}$ such that there is no $\pi \in \mathcal{S}_\omega$ for which $\boldsymbol{a}_\pi \in \mathfrak{S}_\Gamma$ for all $\boldsymbol{a} \in A$. These are the *dual rearrangement numbers*.

**Theorem** *Blass, Brendle, Brian, Hamkins, Hardy, and Larson 2019*

$\mathscr{R}_o \preceq \mathscr{R}_{f,i}$.

*Proof.* The maps $\rho_- : \mathfrak{S}_{cc} \to \mathfrak{S}_{cc}$ the identity and $\rho_+ : \mathcal{S}_\omega \to \mathcal{S}_\omega$ sending $\pi \mapsto \tau_\pi$ form a Tukey connection. □

**Corollary**

$\mathfrak{rr}_o \leq \mathfrak{rr}_{f,i}$ and $\mathfrak{rr}_o^\perp \geq \mathfrak{rr}_{f,i}^\perp$.

# Is every proof dualisable?



We showed $\mathfrak{rr}_o \leq \mathfrak{rr}_{f,i,o}$ in a way that does not translate to a Tukey connection: we used a witness $C \subseteq \mathcal{S}_\omega$ for $\mathfrak{rr}_{f,i,o}$, and created a witness for $\mathfrak{rr}_o$ by considering $C \cup C'$ where $C'$ was the set of mixings of $\pi \in C$ with the identity permutation. But what choice of $\rho_+$ gives us $\rho_+[C] = C \cup C'$?

Instead, we need a **new proof** to show that $\mathfrak{rr}_o^\perp = \mathfrak{rr}_{f,i,o}^\perp$. Let me give a very cursory sketch of the new proof, and refer to my Master's thesis for details.

If $\mathscr{R} = \langle R, X, Y\rangle, \mathscr{S} = \langle S, A, B\rangle$ are relational systems, we can define the *composition* $\mathscr{R} ⌢ \mathscr{S} = \langle T, P, Q\rangle$ where $P = X \times {}^Y A$ and $Q = Y \times B$ and $(x, f)\ T\ (y, b)$ if $x\ R\ y$ and for $f(y) = a$ we have $a\ S\ b$.

# Dualising $\mathfrak{rr}_o = \mathfrak{rr}_{f,i,o}$

**Lemma** *See e.g. Blass 2010*

$\mathfrak{D}(\mathscr{R} \frown \mathscr{S}) = \mathfrak{D}(\mathscr{R}) \cdot \mathfrak{D}(\mathscr{S})$, and

$\mathfrak{B}(\mathscr{R} \frown \mathscr{S}) = \min\{\mathfrak{B}(\mathscr{R}), \mathfrak{B}(\mathscr{S})\}$.

**Theorem** *vdV. 2019, with much help from Brendle*

$\mathfrak{rr}_o^\perp = \mathfrak{rr}_{f,i,o}^\perp$.

*Proof sketch.* We first show that $\mathscr{R}_{f,i,o} \preceq \mathscr{B}$ where $\mathscr{B}$ is a relational system with $\mathfrak{D}(\mathscr{B}) = \mathfrak{b}$. The proof that $\mathfrak{rr}_{f,i,o} \geq \mathfrak{b}$ from the original rearrangement number paper suffices. This proves that $\max\{\mathfrak{rr}_{f,i,o}, \mathfrak{b}\} = \mathfrak{rr}_{f,i,o}$ and $\min\{\mathfrak{rr}_{f,i,o}^\perp, \mathfrak{d}\} = \mathfrak{rr}_{f,i,o}^\perp$.

Then we show that $\mathscr{R}_o \preceq \mathscr{R}_{f,i,o} \frown \mathscr{B}$. By the above, we get $\mathfrak{rr}_o^\perp \geq \mathfrak{rr}_{f,i,o}^\perp$, which is all we need. $\square$

Let $f, g \in {}^\omega\omega$, then we define $f \leq^* g$ (or $g$ *dominates* $f$) if $\{n \in \omega \mid f(n) \not\leq g(n)\}$ is finite.

Let $X, Y \in [\omega]^\omega$, then we define $X \dagger Y$ (or $X$ *splits* $Y$) if $Y \cap X$ and $Y \setminus X$ are both infinite.

Let $\mathcal{M}$ be the $\sigma$-ideal of meagre subsets of ${}^\omega\omega$ and $\mathcal{N}$ be the $\sigma$-ideal of Lebesgue null subsets of ${}^\omega 2$.

$$\mathfrak{d} = \mathfrak{D}(\leq^*, {}^\omega\omega, {}^\omega\omega) \qquad \mathfrak{b} = \mathfrak{B}(\leq^*, {}^\omega\omega, {}^\omega\omega)$$

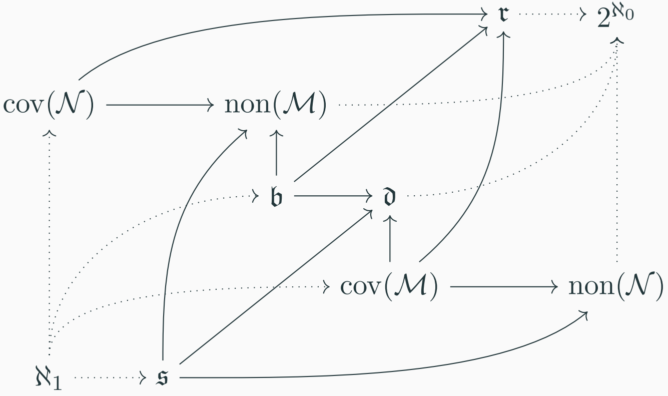
$$\mathfrak{s} = \mathfrak{D}(\dagger, [\omega]^\omega, [\omega]^\omega) \qquad \mathfrak{r} = \mathfrak{B}(\dagger, [\omega]^\omega, [\omega]^\omega)$$

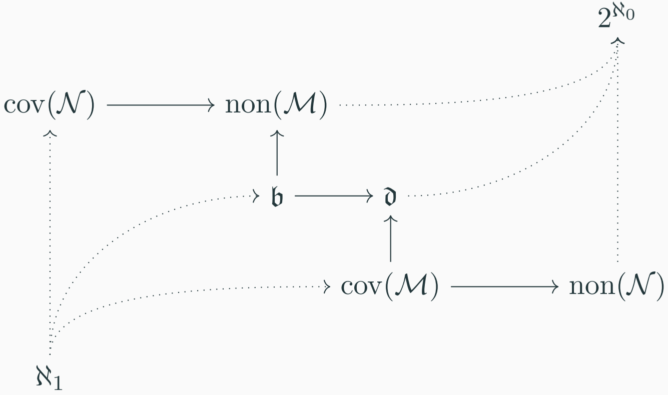
$$\mathrm{cov}(\mathcal{M}) = \mathfrak{D}(\in, {}^\omega\omega, \mathcal{M}) \qquad \mathrm{non}(\mathcal{M}) = \mathfrak{B}(\in, {}^\omega\omega, \mathcal{M})$$

$$\mathrm{cov}(\mathcal{N}) = \mathfrak{D}(\in, {}^\omega 2, \mathcal{N}) \qquad \mathrm{non}(\mathcal{N}) = \mathfrak{B}(\in, {}^\omega 2, \mathcal{N})$$

# Relations between cardinal characteristics






$2^{\aleph_0}$
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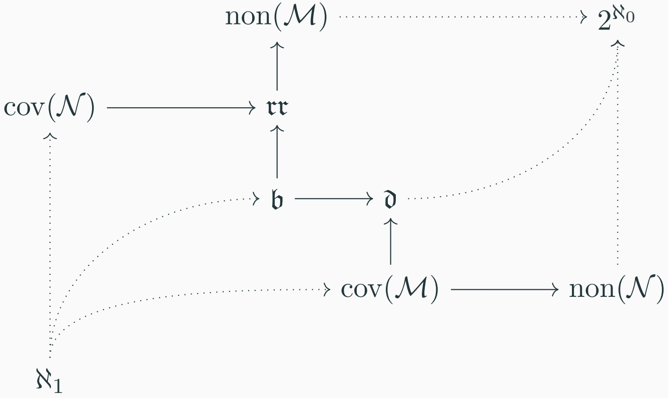
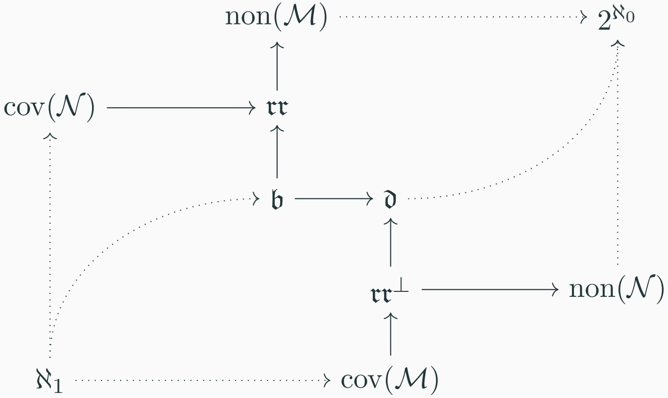
# Overview of rearrangement numbers

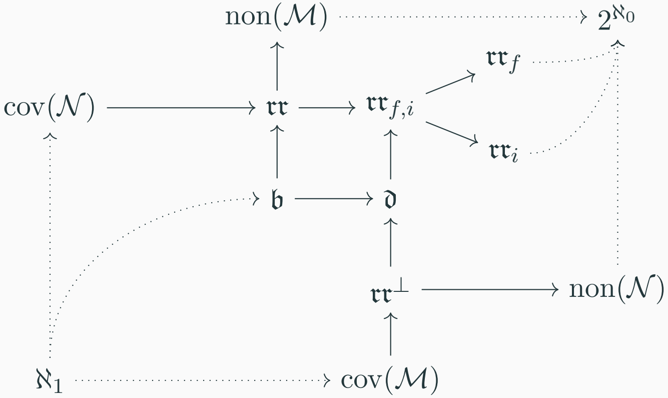




# Overview of rearrangement numbers

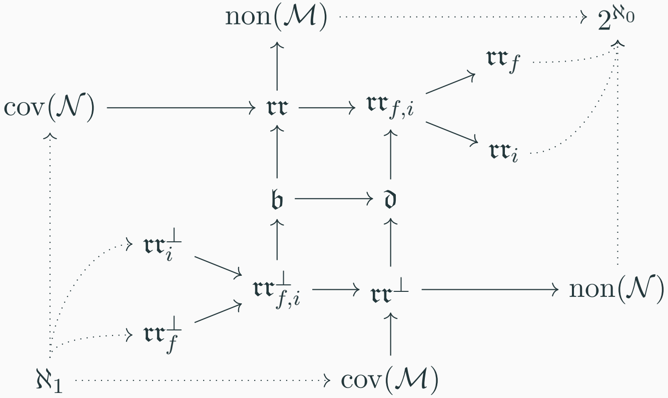


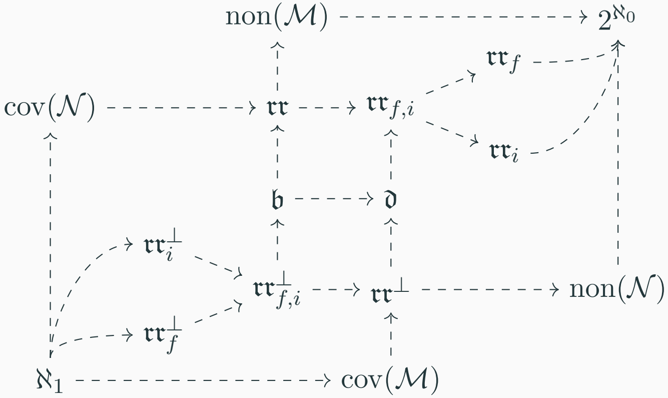
# Overview of rearrangement numbers

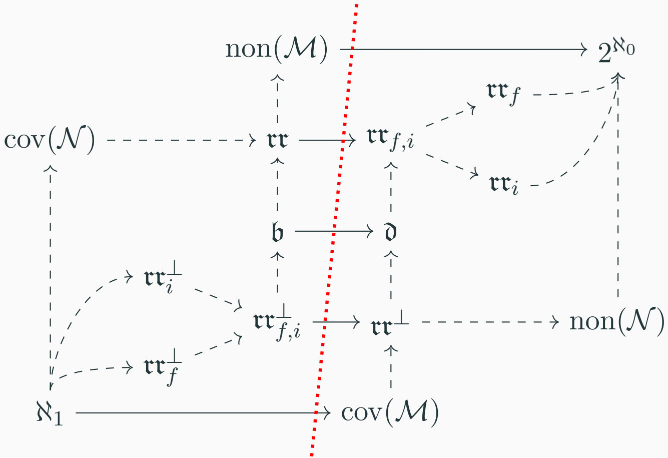


# Overview of rearrangement numbers






non($\mathcal{M}$)
$2^{\aleph_0}$
$\mathfrak{rr}_f$
cov($\mathcal{N}$)
$\mathfrak{rr}$
$\mathfrak{rr}_{f,i}$
$\mathfrak{rr}_i$
$\mathfrak{b}$
$\mathfrak{d}$
$\mathfrak{rr}_i^\perp$
$\mathfrak{rr}_{f,i}^\perp$
$\mathfrak{rr}^\perp$
non($\mathcal{N}$)
$\mathfrak{rr}_f^\perp$
$\aleph_1$
cov($\mathcal{M}$)


$\mathrm{non}(\mathcal{M})$
$2^{\aleph_0}$
$\mathfrak{rr}_f$
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$\mathfrak{rr}_{f,i}^\perp$
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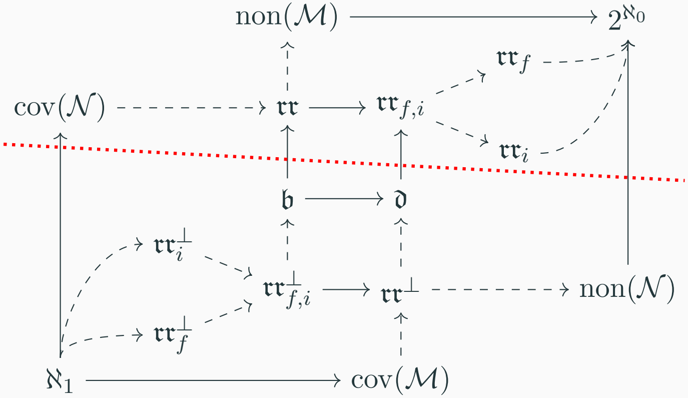
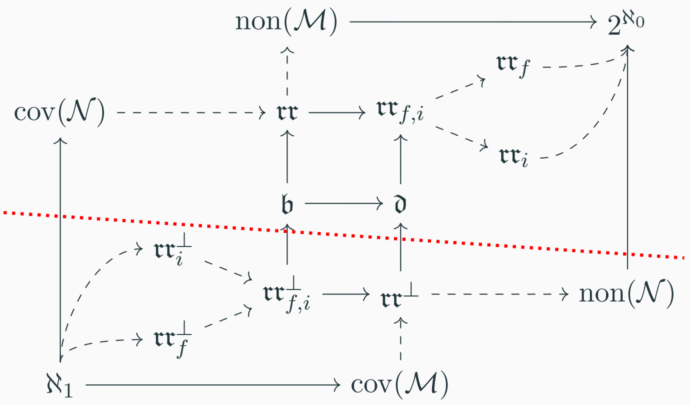
Cohen model

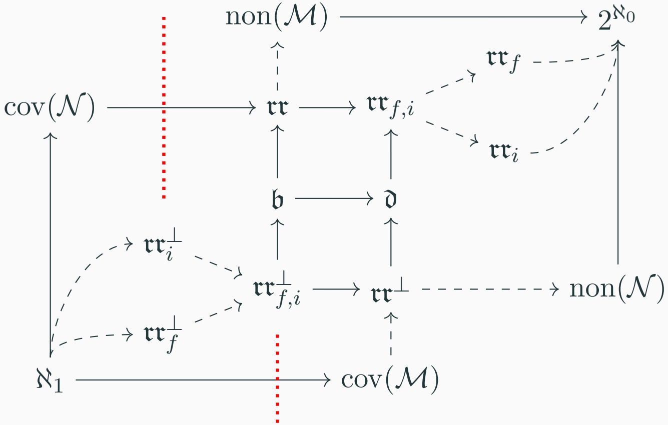



Random model

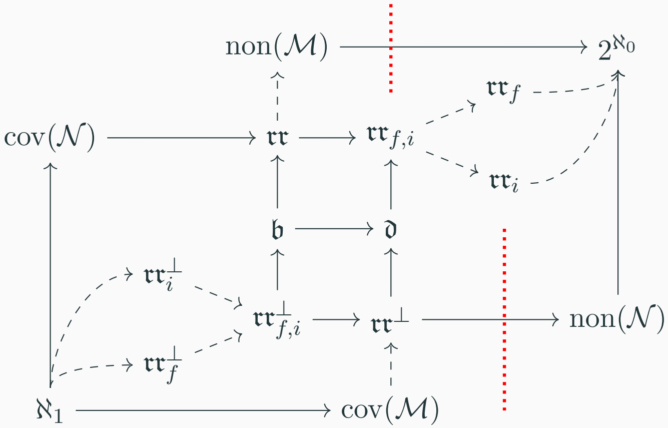



Short random model

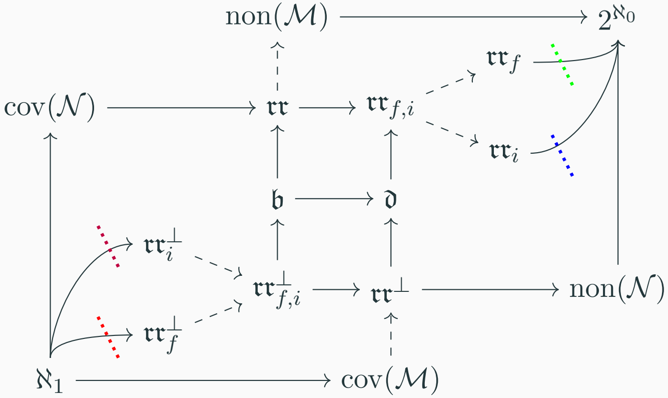
# Consistency results




Hechler model




Short Hechler model




Models using $\mathbb{P}_I$ and $\mathbb{I}_X$

# Open problems



**Question**

Are $\mathfrak{rr}_{fi}$, $\mathfrak{rr}_f$ and $\mathfrak{rr}_i$ consistently different (and similar for the duals)?

**Question**

Are $\mathfrak{rr} < \mathrm{non}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{M}) < \mathfrak{rr}^\perp$ consistent?

**Question**

Are there any cardinal characteristics that form ZFC provable upper bounds to $\mathfrak{rr}_f$ and $\mathfrak{rr}_i$? Or lower bounds to $\mathfrak{rr}_i^\perp$ and $\mathfrak{rr}_f^\perp$?

Consider the least size of a family $\mathcal{X} \subseteq [\omega]^\omega$ such that for every $\boldsymbol{a} \in \mathfrak{S}_{cc}$ there is some $X \in \mathcal{X}$ such that $\sum_X \boldsymbol{a}$ converges.

Clearly $\mathcal{X} = \{\omega\}$ witnesses the above. But what if every $X \in \mathcal{X}$ must be *coinfinite*? This change is significant, and implies that $\mathrm{cov}(\mathcal{M}) \leq |\mathcal{X}|$.

Note that if $X \subseteq \omega$ is cofinite, then $\sum \boldsymbol{a}$ converges / diverges to infinity / oscillates exactly when $\sum_X \boldsymbol{a}$ does. To change the convergence behaviour, we can ignore the cofinite sets.

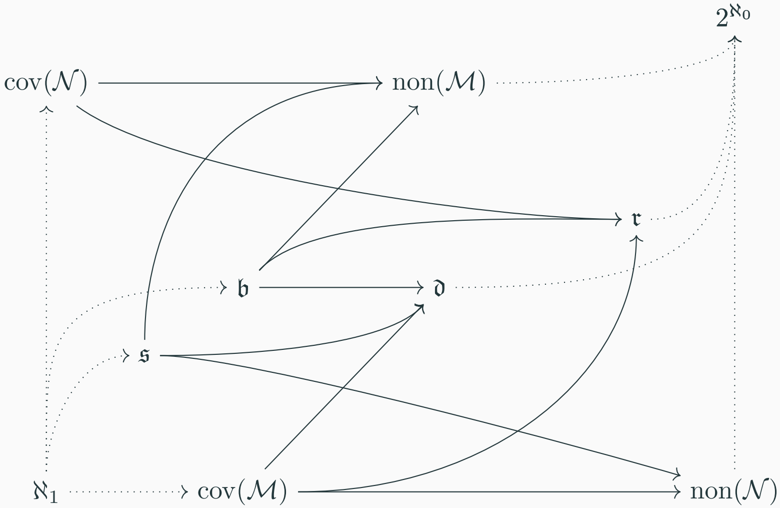
Therefore, we should define subseries numbers using the set $[\omega]^\omega_\omega = \{X \in [\omega]^\omega \mid \omega \setminus X \text{ is infinite}\}$ instead of $[\omega]^\omega$.

For $\Gamma$ a set of convergence behaviours, we define the relation $S_\Gamma \subseteq \mathfrak{S}_{cc} \times [\omega]^\omega_\omega$ by $\boldsymbol{a}\ S_\Gamma\ X$ if and only if $\sum_X \boldsymbol{a}$ behaves according to $\Gamma$. We define $\mathfrak{ß}_\Gamma = \mathfrak{D}(S_\Gamma, \mathscr{S}_{cc}, [\omega]^\omega_\omega)$.

Our main interests will be $\mathfrak{ß} = \mathfrak{ß}_{i,o}$ , $\mathfrak{ß}_i$ , $\mathfrak{ß}_o$ , $\mathfrak{ß}_c$ , $\mathfrak{ß}_{cc}$ and $\mathfrak{ß}_{ac}$ and their dual cardinal characteristics.

The subseries numbers $\mathfrak{ß}$, $\mathfrak{ß}_i$ and $\mathfrak{ß}_o$ were originally studied by Brendle, Brian, and Hamkins (2019) and defined using $[\omega]^\omega$. It is, however, easy to prove that our definition results in the same cardinal characteristics.

The cardinals $\mathfrak{ß}_c$ , $\mathfrak{ß}_{cc}$ and $\mathfrak{ß}_{ac}$ were introduced in my Master's thesis and are subject of the preprint vdV. (2025).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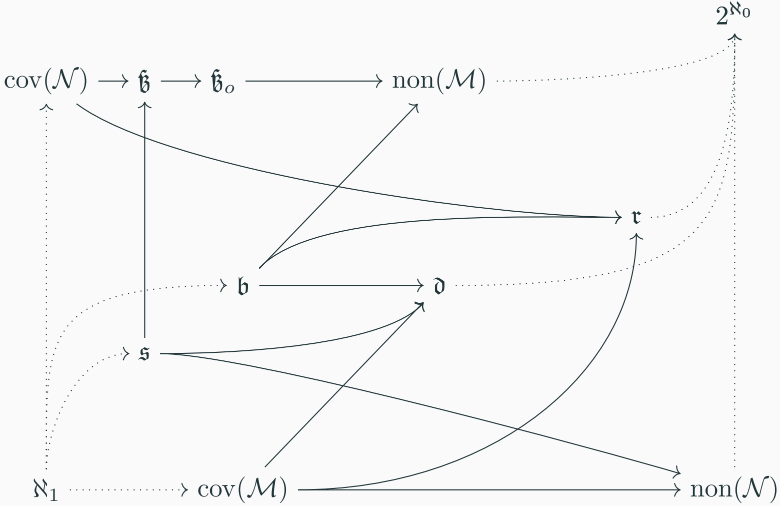
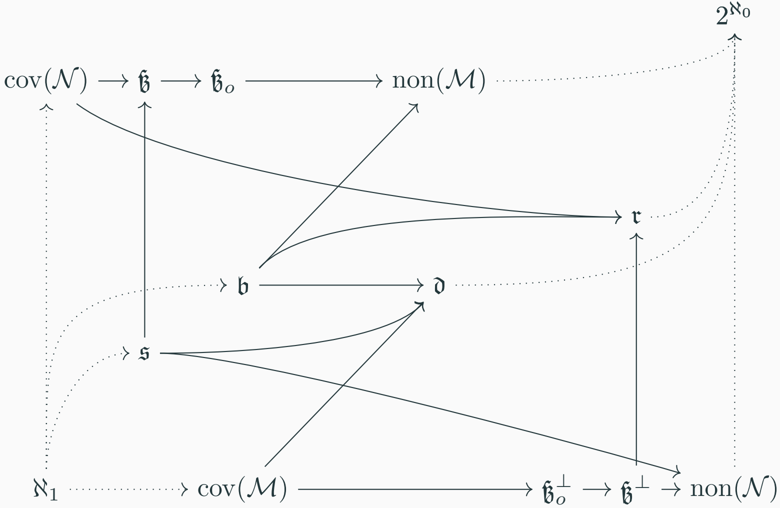
# Overview of subseries numbers





$2^{\aleph_0}$
$\mathrm{cov}(\mathcal{N})$
$\mathfrak{ß}$
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$\mathrm{non}(\mathcal{M})$
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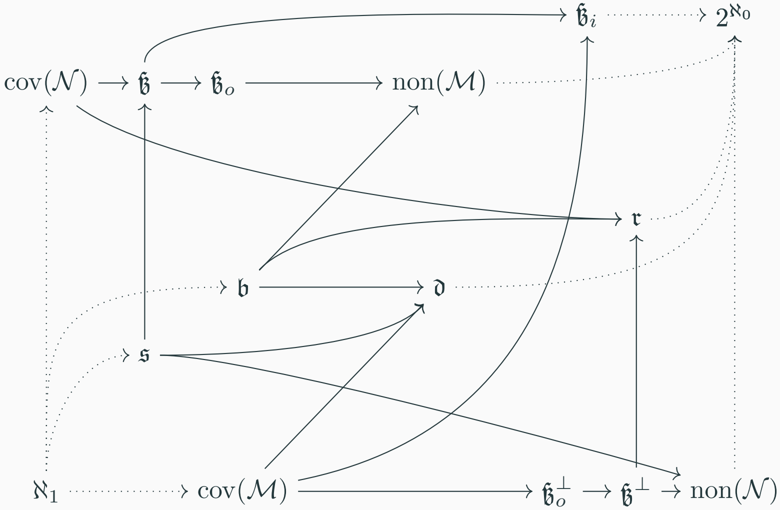
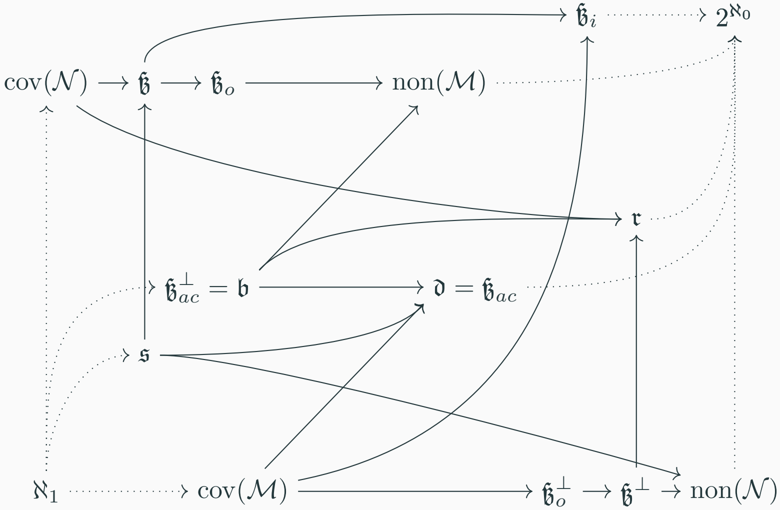
$\mathrm{non}(\mathcal{N})$

# Overview of subseries numbers

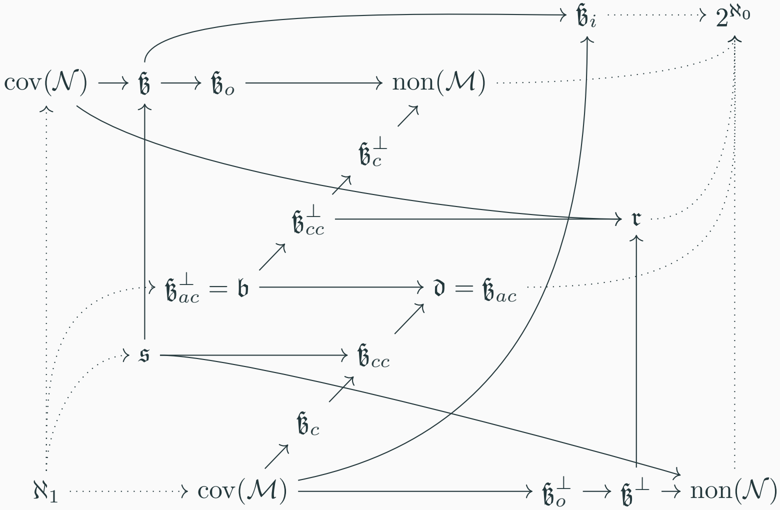


# Overview of subseries numbers

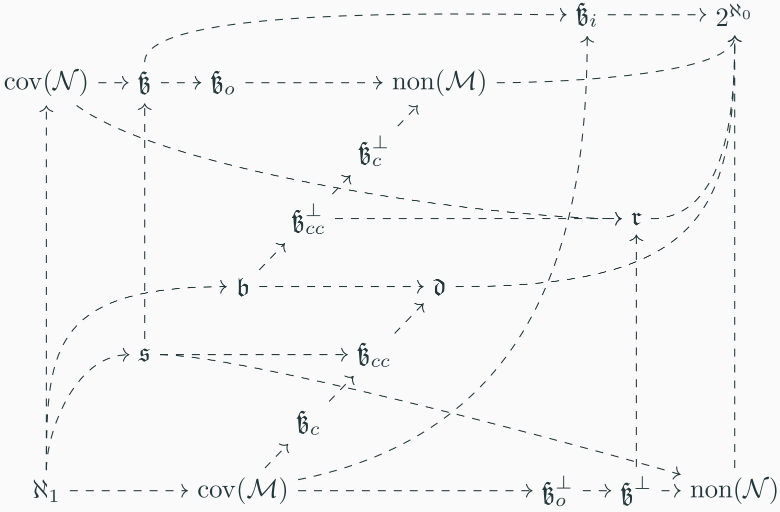


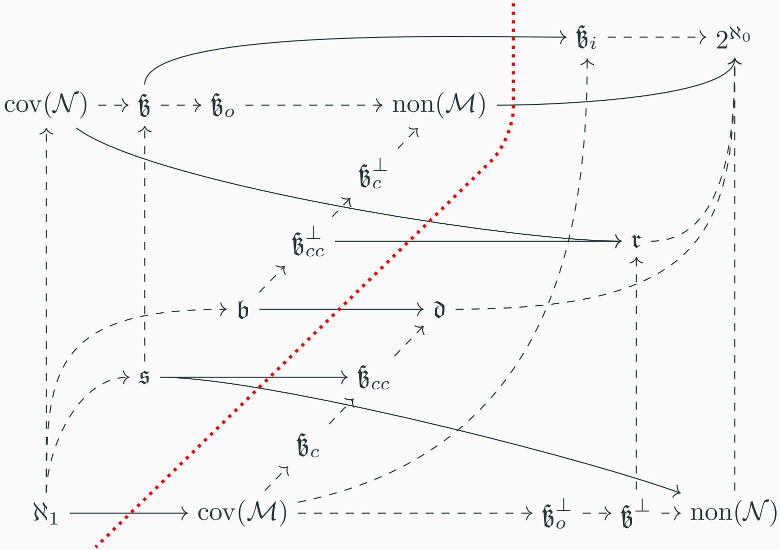
# Overview of subseries numbers





# Consistency results




$\mathfrak{h}_i$
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$\mathrm{cov}(\mathcal{N})$
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$\mathfrak{h}_c^\perp$
$\mathfrak{h}_{cc}^\perp$
$\mathfrak{r}$
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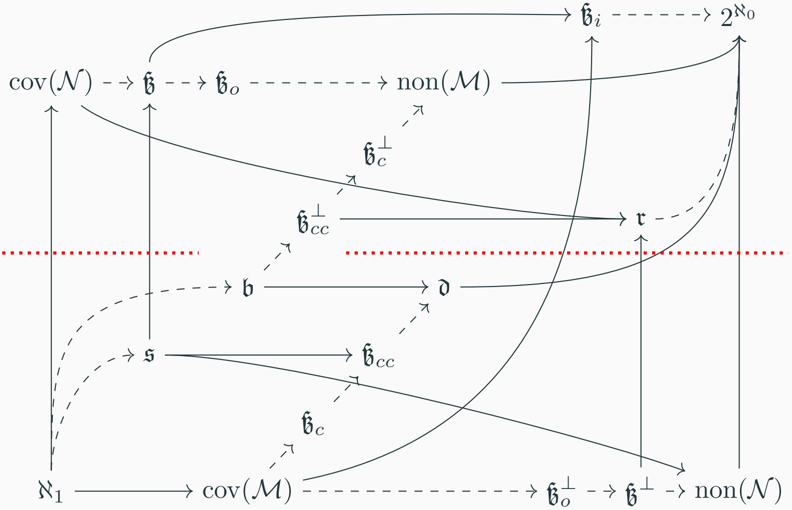
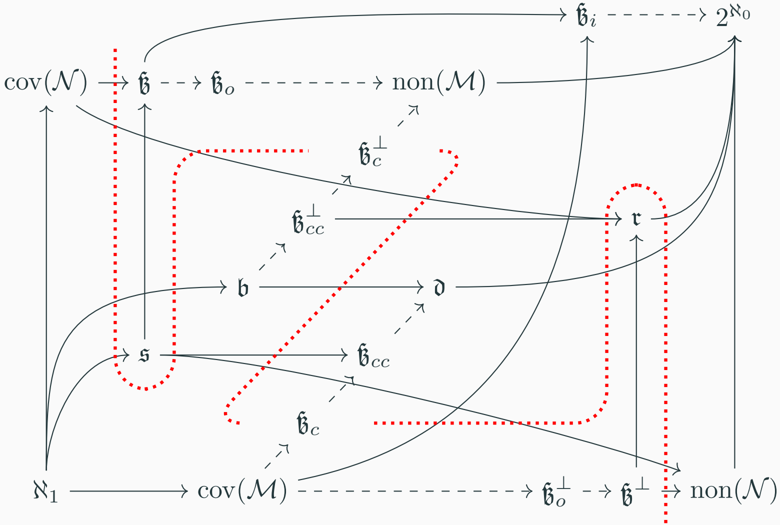
$\mathfrak{s}$
$\mathfrak{h}_{cc}$
$\mathfrak{h}_c$
$\aleph_1$
$\mathrm{cov}(\mathcal{M})$
$\mathfrak{h}_o^\perp$
$\mathfrak{h}^\perp$
$\mathrm{non}(\mathcal{N})$

Cohen model

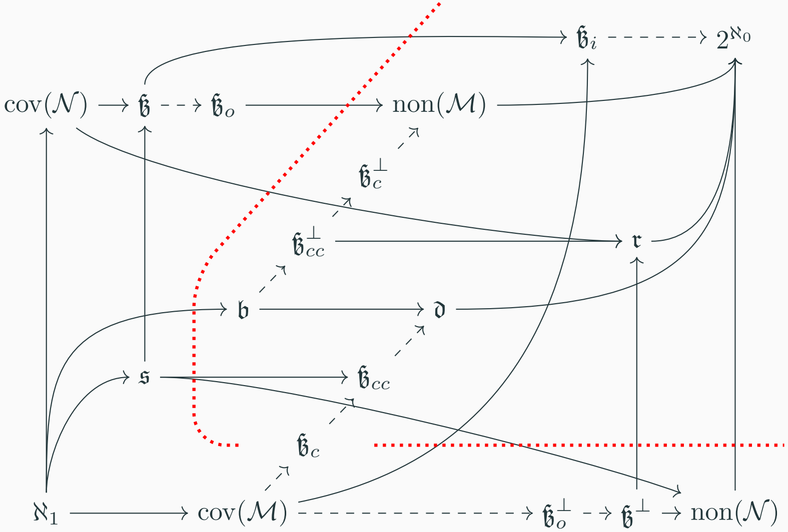



Random model

# Consistency results




Blass-Shelah model



$2^{\aleph_0}$
$\mathfrak{z}_i$
$\text{cov}(\mathcal{N})$
$\mathfrak{z}$
$\mathfrak{z}_o$
$\text{non}(\mathcal{M})$
$\mathfrak{z}_c^\perp$
$\mathfrak{z}_{cc}^\perp$
$\mathfrak{r}$
$\mathfrak{b}$
$\mathfrak{d}$
$\mathfrak{s}$
$\mathfrak{z}_{cc}$
$\mathfrak{z}_c$
$\aleph_1$
$\text{cov}(\mathcal{M})$
$\mathfrak{z}_o^\perp$
$\mathfrak{z}^\perp$
$\text{non}(\mathcal{N})$


Laver model

# The dual to $\mathfrak{z}_i$



Remember that $\mathfrak{z}_i^\perp$ is the least number of a set $A \subseteq \mathfrak{S}_{cc}$ such that there exists no $X \in [\omega]^\omega_\omega$ for which $\sum_X \boldsymbol{a}$ tends to infinity for all $\boldsymbol{a} \in A$.

It is easy to see that $\mathfrak{z}_i^\perp > 2$. Consider for $\boldsymbol{a}, \boldsymbol{b} \in \mathfrak{S}_{cc}$ the sets

| | |
|---|---|
| $X^{++} = \{n \mid a_n > 0, b_n > 0\}$ | $X^{+-} = \{n \mid a_n > 0, b_n \leq 0\}$ |
| $X^{-+} = \{n \mid a_n \leq 0, b_n > 0\}$ | $X^{--} = \{n \mid a_n \leq 0, b_n \leq 0\}$ |

At least one cell per row gives a subseries of $\sum \boldsymbol{a}$ tending to infinity.
At least one cell per column gives a subseries of $\sum \boldsymbol{b}$ tending to infinity.

A case-by-case analysis shows we can make both $\sum \boldsymbol{a}$ and $\sum \boldsymbol{b}$ tend to infinity with one subset.

# The dual to $\mathfrak{z}_i$

Will Brian (2018) showed that $\mathfrak{z}_i^\perp > 3$, using a more complicated case-by-case argument. Surprisingly, Fedor Nazarov showed on *MathOverflow* (and Brian repeated the argument in his paper) that:

**Theorem** *Nazarov*

$\mathfrak{z}_i^\perp = 4$ .

That is, there exist four CC series such that for any $X \in [\omega]^\omega_\omega$, at least one of the four series will diverge by oscillation (and thus not tend to infinity).

For this reason, it is hard to build a forcing notion that forces $\mathfrak{z}_i < \mathfrak{c}$. Indeed, the consistency of $\mathfrak{z}_i < \mathfrak{c}$ is an open problem.

Remember that $\mathfrak{rr}_o = \mathfrak{rr}_{i,o}$. Can we prove $\mathfrak{B}_o = \mathfrak{B}_{i,o}$? This is also an open problem, but there is a partial solution.

**Theorem** *Brendle, Brian, and Hamkins 2019, vdV. 2019*

$\mathfrak{B}_o \leq \max\{\mathfrak{B}, \mathfrak{b}\}$ and $\mathfrak{B}_o^\perp \geq \min\{\mathfrak{B}^\perp, \mathfrak{d}\}$.

As with the rearrangement number, the original proof cannot be translated to a Tukey connection. However, there exists a Tukey connection with two sequential compositions that shows that $\mathfrak{B}_o \leq \max\{\mathfrak{B}, \mathfrak{b}, \mathfrak{s}\}$. Since $\mathfrak{s} \leq \mathfrak{B}$ (and dually $\mathfrak{r} \geq \mathfrak{B}^\perp$), this provides a dualisable proof of the above theorem.

**Theorem** *vdV. (2025+)*

$\mathscr{S} = \langle \dagger, [\omega]^\omega_\omega, [\omega]^\omega_\omega \rangle \quad \preceq \quad \mathscr{S}_{cc} = \langle S_{cc}, \mathfrak{S}_{cc}, [\omega]^\omega_\omega \rangle$.

*Proof.* We let $\rho_+ : [\omega]^\omega_\omega \to [\omega]^\omega_\omega$ be the identity. Given $X \in [\omega]^\omega_\omega$ assume without loss that $0 \in X$, and let $\langle I_n \mid n \in \omega \rangle$ be an interval partition of $\omega$ such that $X = \bigcup_{n\in\omega} I_{2n}$. For $i \in \omega$ let $n \in \omega$ such that $i \in I_n$ and $s_n = |I_n|$, then we define $a_i = \frac{(-1)^n}{s_n \cdot n}$, and see that $\boldsymbol{a} \in \mathfrak{S}_{cc}$. We let $\rho_-(X) = \boldsymbol{a}$.

Let $Y \in [\omega]^\omega_\omega$ and $\sum_Y \boldsymbol{a}$ be CC. If $Y \cap X$ is finite, then $a_i > 0$ for finitely many $i \in Y$, contradicting that $\sum_Y \boldsymbol{a}$ is CC. If $Y \setminus X$ is finite, then $a_i < 0$ for finitely many $i \in Y$, also a contradiction. Thus $X$ splits $Y$. □

**Question**

Is $\mathfrak{b}_i = \mathfrak{c}$ provable?

**Question**

Is $\mathfrak{b} = \mathfrak{b}_o$ or $\mathfrak{b}^\perp = \mathfrak{b}_o^\perp$ provable?

**Question**

Is $\mathrm{cov}(\mathcal{M}) < \mathfrak{b}_o^\perp$ consistent?

**Question**

Are any of $\mathrm{cov}(\mathcal{M}) < \mathfrak{b}_c$ or $\mathfrak{b}_c < \mathfrak{b}_{cc}$ or $\mathfrak{b}_{cc} < \mathfrak{d}$ consistent?

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