Labelled Sets

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Let κ be an infinite cardinal. Consider the poset $[\kappa]^2 := \{\langle\alpha, \beta\rangle : \alpha < \beta < \kappa\}$ ordered by $\langle\alpha, \beta\rangle \leq \langle\gamma, \delta\rangle$ iff $\alpha \leq \gamma$ and $\beta \leq \delta$.

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Posets with no infinite antichains are sometimes called FAC posets.

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2. *If ν is an uncountable limit cardinal, then $cov(P) \geq \nu$ if and only if either P or P^* contains a partial order Q of the form $\sum_{a \in C} Q_a$, where C is a chain of cardinality $cf(\nu)$, $Q_a \cong [\kappa_a^+]^2$ and $\langle \kappa_a : a \in C \rangle$ is a family of pairwise distinct cardinals that satisfies $\sup_{a \in C} \kappa_a = \nu$.*

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Recall that, if $\mathcal{A}$ is a class of structures, a *basis* for $\mathcal{A}$ is a subclass $\mathcal{B} \subseteq \mathcal{A}$ such that for every $A \in \mathcal{A}$ there exists $B \in \mathcal{B}$ with $B \leq A$, i.e. B embeds into A.

Suppose $P = \sum_{x \in X} [\kappa_x]^2$ and $Q = \sum_{y \in Y} [\lambda_y]^2$, with X and Y chains and $x \mapsto \kappa_x$, $y \mapsto \lambda_y$ injective.

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$P \leq Q$ if and only if there exists a weakly order preserving map $\varphi : X \to Y$ such that, for all $y \in Y$, $\sum_{x \in \varphi^{-1}\{y\}} \kappa_x \leq \lambda_y$. Here, $\sum$ is the lexicographic sum and $\leq$ denotes embeddability.

A word on the proof

Key point: if $[\kappa]^2 = A \cup B$ with $A \times B \neq \emptyset$ and $a < b$ for all $a \in A$ and $b \in B$, then $A = \{\langle 0, 1\rangle\}$ or $A = \{\langle 0, 1\rangle, \langle 0, 2\rangle\}$.

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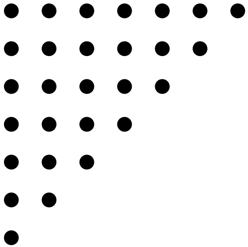

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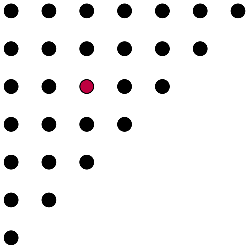

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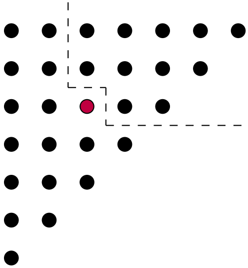

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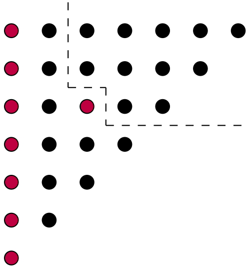

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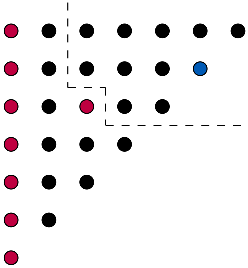

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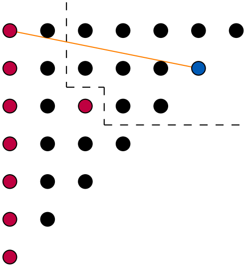

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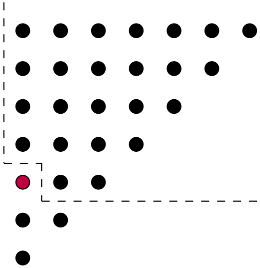

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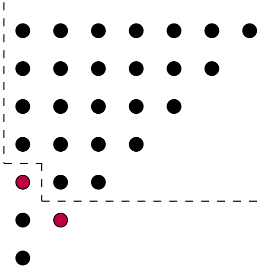

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If $\psi : \sum_{x \in X} [\kappa_x]^2 \to \sum_{y \in Y} [\lambda_y]^2$ is an embedding, let

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By the key point, $|E_\psi(x)| \leq 3$.
The map $\varphi : X \to Y$ defined by $\varphi(x) := \max(E_\psi(x))$ has the desired properties.

Returning to the sufficient condition, by composing the map $x \mapsto \kappa_x$ with $\aleph^{-1}$, we are led to consider the following category:

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Goal: Extend the theory of embeddability of uncountable linear orderings to the labelled context.

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Recall that a set $A \subseteq \mathbb{R}$ is $\aleph_1$-*dense* iff it meets every non-empty interval of $\mathbb{R}$ in $\aleph_1$-many points.

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In any such model, the family of posets of the form $\sum_{x \in X}[\kappa_x]^2$ with $X \subseteq \mathbb{R}$ uncountable and $x \mapsto \kappa_x$ injective has a basis of size 1.

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To fix this, we need to find a subposet $\mathbb{P}^*$ of $\mathbb{P}$ which is sufficiently rich (it has enough dense sets), but is not too big (it has the ccc).

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Theorem

Assume CH. *Let* (A, f) *and* (B, g) *be labelled sets with* A *and* B $\aleph_1$*-dense sets of reals. Then there is a ccc poset which forces an embedding* $(A, f) \leq (B, g)$.

Thank you :)