Does $\mathcal{P}(\omega)/\mathrm{Fin}$ know its right hand from its left?

Part 3

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Back-and-forth again with the Lifting Lemma

This talk focuses on three further consequences of the Lifting Lemma and the main theorem presented in the previous talks, as well as some related open questions.

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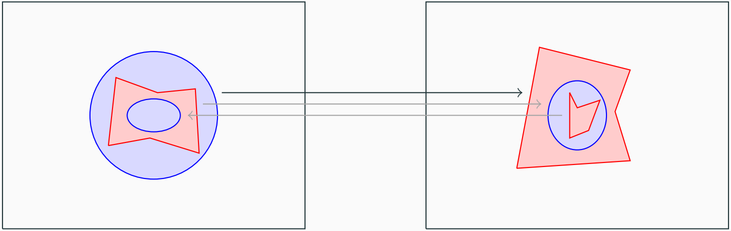
Recall that the Lifting Lemma can be used in a back-and-forth argument to prove, assuming CH, that σ and σ^{-1} are conjugate.

$\left(\mathcal{P}(\omega)/\mathrm{Fin}, \sigma\right)$ $\left(\mathcal{P}(\omega)/\mathrm{Fin}, \sigma^{-1}\right)$



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In fact, the argument we gave shows something a little stronger:

Theorem

Suppose $\langle \mathbb{A}, \sigma^{-1} \rangle$ *is a countable elementary substructure of* $\langle {}^{\mathcal{P}(\omega)}/_{\mathrm{Fin}}, \sigma^{-1} \rangle$*, and* η *is an embedding of* $\langle \mathbb{A}, \sigma^{-1} \rangle$ *into* $\langle {}^{\mathcal{P}(\omega)}/_{\mathrm{Fin}}, \sigma \rangle$*. Then there is an isomorphism* ϕ *from* $\langle {}^{\mathcal{P}(\omega)}/_{\mathrm{Fin}}, \sigma^{-1} \rangle$ *to* $\langle {}^{\mathcal{P}(\omega)}/_{\mathrm{Fin}}, \sigma \rangle$ *with* $\phi \restriction \mathbb{A} = \eta$*.*

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Corollary

There is a nontrivial automorphism of ${}^{\mathcal{P}(\omega)}/_{\mathrm{Fin}}$ *that commutes with* σ*, i.e., an automorphism* ϕ *such that* $\phi \circ \sigma = \sigma \circ \phi$*.*

Square roots of trivial mappings

Let f denote some permutation of ω with exactly one $\mathbb{Z}$-like orbit.

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

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If σ and σ^{-1} are conjugate, there is a (necessarily nontrivial) automorphism ϕ of $\mathcal{P}(\omega)/\mathrm{Fin}$ such that $\phi \circ \phi = \alpha_f$. Furthermore, some such nontrivial automorphism ϕ is conjugate to σ.

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Proof: Let f denote a $\mathbb{Z}$-like orbit on ω as before:

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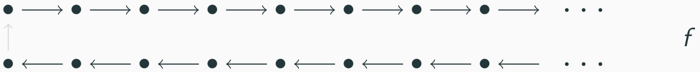

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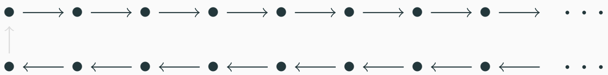

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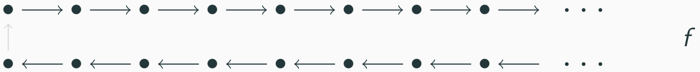

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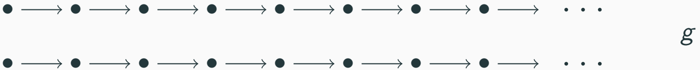

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Thus we get two consequences of CH regarding the shift map:

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$$\mathrm{Ind}(f) = |\mathrm{domain}(f) \setminus \mathrm{image}(f)| - |\mathrm{image}(f) \setminus \mathrm{domain}(f)|.$$

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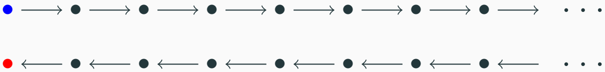

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In particular, it makes sense to write $\mathrm{Ind}(\alpha_f)$, not just $\mathrm{Ind}(f)$. For example, $\mathrm{Ind}(\sigma) = 1$ and $\mathrm{Ind}(\sigma^{-1}) = -1$.

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In other words, the index parity is a ZFC obstruction to conjugacy.

Theorem (B. and Farah, 2024)

Let α and β be trivial automorphisms of ${}^{\mathcal{P}(\omega)}/_{\mathrm{Fin}}$. TFAE:

1. *α and β are conjugate in a forcing extension.*
2. *CH proves α and β are conjugate.*
3. *$\mathrm{Ind}(\alpha)$ and $\mathrm{Ind}(\beta)$ have the same parity, and the structures $\langle {}^{\mathcal{P}(\omega)}/_{\mathrm{Fin}}, \alpha \rangle$ and $\langle {}^{\mathcal{P}(\omega)}/_{\mathrm{Fin}}, \beta \rangle$ are elementarily equivalent.*

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Suppose α *and* β *are trivial automorphisms of* $\mathcal{P}(\omega)/\mathrm{Fin}$ *and the structures* $\langle \mathcal{P}(\omega)/\mathrm{Fin}, \alpha \rangle$ *and* $\langle \mathcal{P}(\omega)/\mathrm{Fin}, \beta \rangle$ *are elementarily equivalent. Does this imply* α *and* β *have the same index parity?*

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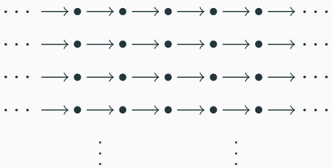
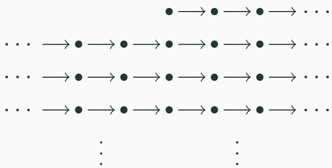
Open Question

Let f be a permutation of ω with infinitely many $\mathbb{Z}$-like orbits, and let g be an almost permutation with infinitely many $\mathbb{Z}$-like orbits and one $\mathbb{N}$-like orbit. Is $\langle \mathcal{P}(\omega)/\mathrm{Fin}, \alpha_f \rangle \equiv \langle \mathcal{P}(\omega)/\mathrm{Fin}, \alpha_g \rangle$?





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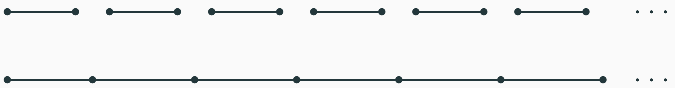

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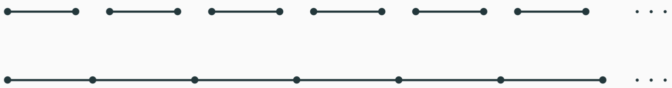

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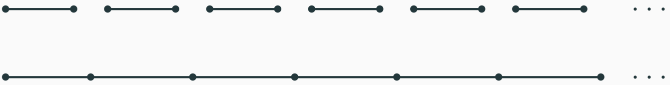

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Just as $\mathbb{H}$ can be obtained from $\mathbb{M}$ by gluing some points together, there is an equivalence relation $\sim$ on $\mathbb{M}^*$ such that $\mathbb{H}^* = \mathbb{M}^*/\sim$.

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$\mathbb{H}^*$ is obtained from $\mathbb{M}^*$ by gluing these I_u together, the right endpoint of I_u being glued to the left endpoint of $I_{\sigma(u)}$. Each of these I_u is called a *standard subcontinuum* of $\mathbb{H}^*$.

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Theorem (Vignati, 2021)

$\mathrm{OCA} + \mathrm{MA}$ *implies all self-homeomorphisms of* $\mathbb{H}^*$ *are trivial, and in particular there is no order-reversing self-homeomorphism of* $\mathbb{H}^*$.

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Using the theorem on the previous slide (and using CH again), there is an order-preserving self-homeomorphism $F : \mathbb{M}^* \to \mathbb{M}^*$ such that $\pi \circ F = f$.

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Furthermore, because G sends each I_u to itself (only reversed), H maps each I_u to $I_{f(u)}$, just like F.

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Because H preserves the equivalence classes of $\sim$, the function $[x]_\sim \mapsto [H(x)]_\sim$ is a well-defined mapping on $\mathbb{H}^*$.

A sketch of the proof

Theorem (B., Dow, and Hart, 2024)
CH *implies there is an order-reversing self-homeomorphism of* $\mathbb{H}^*$*.*

Proof sketch, assuming the two main ingredients:

Recall $\mathbb{H}^* = \mathbb{M}^*/\sim$, where the equivalence classes of $\sim$ are either singletons or the sets of the form $\{\bar{1}_u, \bar{0}_{\sigma(u)}\}$. Observe that

- $G(\bar{1}_u) = \bar{0}_u$ and $G(\bar{0}_{\sigma(u)}) = \bar{1}_{\sigma(u)}$, and
- $F(\bar{0}_u) = \bar{0}_{f(u)}$ and $F(\bar{1}_{\sigma(u)}) = \bar{1}_{f\circ\sigma(u)} = \bar{1}_{\sigma^{-1}\circ f(u)}$.

Thus $H = F \circ G$ maps the set $\{\bar{1}_u, \bar{0}_{\sigma(u)}\}$ to the set $\{\bar{0}_{f(u)}, \bar{1}_{\sigma^{-1}(f(u))}\}$, which is also an equivalence class of $\sim$.

Because H preserves the equivalence classes of $\sim$, the function $[x]_\sim \mapsto [H(x)]_\sim$ is a well-defined mapping on $\mathbb{H}^*$. This function is the sought-after order-reversing self-homeomorphism of $\mathbb{H}^*$.

A few more questions

Open Question

Is it consistent with $\neg$CH *that* σ *and* σ^{-1} *are conjugate?*

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Assuming CH, *suppose there is a nontrivial automorphism* α *of* $\mathcal{P}(\omega)/\mathrm{Fin}$ *such that* $\langle \mathcal{P}(\omega)/\mathrm{Fin}, \alpha \rangle \equiv \langle \mathcal{P}(\omega)/\mathrm{Fin}, \sigma \rangle$. *Does this imply* α *is conjugate to* σ*?*

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Open Question (Szeptycki)

Is it consistent (does it follow from CH*) to have a somewhere trivial automorphism of* $\mathcal{P}(\omega)/\mathrm{Fin}$ *that conjugates* σ *and* σ^{-1}*?*

A few more questions

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Is it consistent (does it follow from CH*) to have a somewhere trivial automorphism of* $\mathcal{P}(\omega)/\mathrm{Fin}$ *that conjugates* σ *and* σ^{-1}*?*

Open Question (Moore)

Can we characterize when CH *implies two structures of the form* $\langle \mathcal{P}(\omega)/\mathrm{Fin}, \alpha, \beta \rangle$ *are isomorphic?*