Remote Points and Extremal Disconnectedness

2011 Winterschool, Hejnice

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Further, Frolik showed the following:

Theorem. *(Frolik) If X is not pseudocompact, then X^* is not homogeneous.*

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Theorem. *(Fine and Gilman, Dow) Under CH, separable spaces have remote points and it is consistent that there is a separable space with no remote points.*

Using Remote Points

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Corollary. *If $\{X_n : n < \omega\}$ are separable spaces, then TFAE:*

- $\prod_{n<\omega} X_n$ *is pseudocompact,*
- $\beta(\prod_{n<\omega} X_n) = \prod_{n<\omega} \beta X_n$,
- $\beta(\prod_{n<\omega} X_n)$ *is homeomorphic to* $\prod_{n<\omega} \beta X_n$.

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How can ED help the study of remote points?

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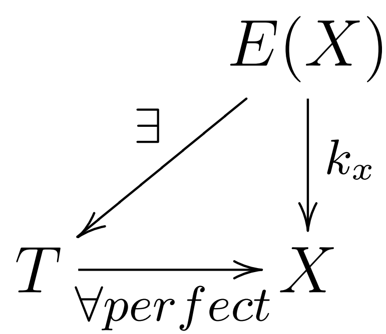

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So basically, **for compact spaces**, the absolute is determined by the boolean algebra of regular open subsets.

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Since X is compact, EX has **all** ultrafilters in $s(RO({}^{\omega}2))$. The result follows from this. $\square$

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Consider the function $k_{\beta X} : \beta E(X) \to \beta X$, let $T' = k^{\leftarrow}_{\beta X}[T(X)]$ and restrict $k' = k_{\beta X} \restriction_{T'} : T' \to T(X)$

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The function $k' : T' \to T(X)$ turns out to be a homeomorphism (by the properties of ED spaces and absolutes). Also, $T' \subset T(EX)$.

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Corollary. *If X and Y are normal coabsolute spaces, then $T(X)$ is homeomorphic to $T(Y)$.*

This is not the only way to obtain spaces with the same space of remote points: for example $T(\mathbb{Q})$ and $T(\mathbb{Q} \oplus K)$ are homeomorphic whenever K is compact.

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Theorem. *(Woods) Let κ be an infinite cardinal and C_κ be a free sum of κ copies of the Cantor set ${}^\omega 2$. If X is a locally compact, non-compact metrizable space with no isolated points and $w(X) = \kappa$, then there is a continuous, perfect and irreducible surjection $f : C_\kappa \to X$. Thus, $E(X) \approx E(C_\kappa)$ and $T(X) \approx T(C_\kappa)$.*

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By the way, it is easy to see that $T(C_\kappa)$ is dense in C^*_κ.

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Theorem. *Let X be a metrizable space.*

- $E(X) \approx E({}^\omega\omega)$ *if and only if* X *is completely metrizable, separable and nowhere locally compact.*
- *Let* $\kappa > \omega$. *Then* $E(X) \approx E({}^\omega\kappa)$ *if and only if* X *is completely metrizable and for each non-empty open set* $U \subset X$, *we have that* $w(U) = \kappa$.

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(4) besides this method, how can we show that spaces of remote points are (non) homeomorphic?

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This question has not been answered, to my knowledge.

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