Topological fractals (and maybe also self regenerating fractals)

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Joint work with Benjamin Vejnar, paper in preparation

Peano continua

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- A metric space is a Peano continuum iff it is a continuous image of $I = [0, 1]$,
- A continuum X is a Peano continuum iff it has the property S, i.e. iff for every $\varepsilon > 0$ there exists a finite cover of X formed by connected sets with diameter smaller than ε.

Self–similar sets

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We say that a nonempty compact $X \subseteq \mathcal{R}^n$ is self–similar if there exist similarities $f_1, f_2, \ldots, f_n : X \to X$ with ratio strictly less than 1 such that $X = f_1(X) \cup \cdots \cup f_n(X)$.

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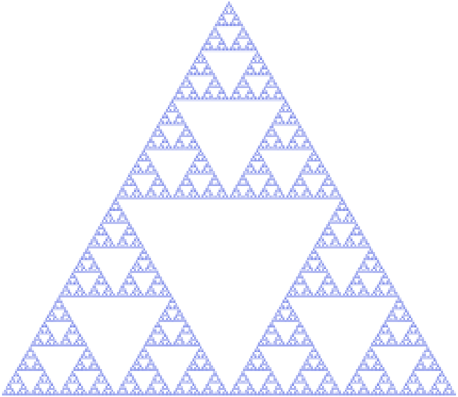
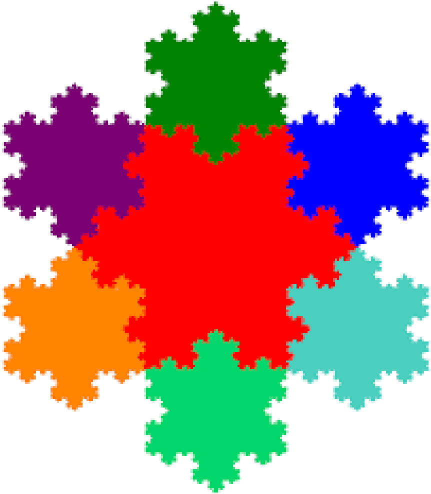
- Sierpinski triangle, Koch snowflake
- The circle and the Koch curve are **not** self-similar
- Every connected self–similar set is a Peano continuum

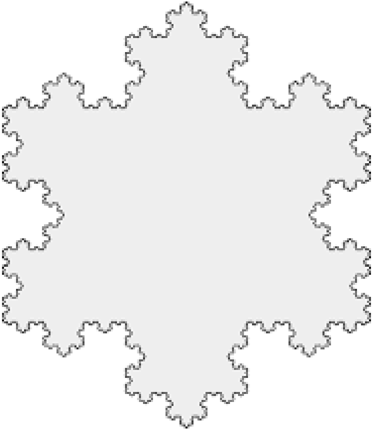
Sierpinski triangle



Koch snowlake as a union of smaller Koch snowlakes



Koch curve



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- $[0, 1]^n$ is a metric fractal for every natural n, but $[0, 1]^{\mathbb{N}}$ is not a metric fractal

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- All metric fractals are of a finite topological dimension
- Hata's conjecture (1985): Every Peano continuum of a finite topological dimension is a metric fractal
- There exists a one dimensional plane Peano continuum that is not even homeomorphic to any metric fractal (T. Banakh, M. Nowak; 2013)

Topological fractals, Hata's conjecture

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We say that a nonempty compact metric space X is a topological fractal if there exist continuous functions $f_1, \dots, f_n : X \to X$ satisfying $f_1(X) \cup \dots \cup f_n(X) = X$ such that for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that for every $1 \leq i_1, i_2, \dots, i_k \leq n$ is the diameter of the set $f_{i_1} f_{i_2} \dots f_{i_k}(X)$ less than ε.

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- Again every connected topological fractal has a property S and thus is a Peano continuum
- Hata's conjecture (unsolved since 1985): Every Peano continuum is a topological fractal

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- $[0, 1]^n$ is a topological fractal for every natural n and moreover, two maps suffice, but it is not known whether $[0, 1]^{\mathbb{N}}$ is a topological fractal
- For every Peano continuum X, glueing to X an arc yields a topological fractal (Dumitru 2011)
- Every Peano continuum that contains a self–regenerating fractal as a subset with nonempty interior is a topological fractal (Nowak 2021)

The main result

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This is the optimal result for all non–degenerate spaces.

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Every Peano continuum with uncountably many local cut points is a topological fractal. Moreover, there is a structure of a topological fractal consisting of just three maps.

Self regenerating fractals

Definition

We say that a nonempty compact metric space X is a self regenerating fractal if for every nonempty open set $U \subseteq X$ there exist continuous functions $f_1, \ldots, f_n : X \to X$ constant on the complement of U that witness that X is a topological fractal.

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Theorem (M. Nowak, 2021)

Every Peano continuum containing a self regenerating fractal as a subset with nonempty interior is a topological fractal itself.

Theorem

If every Peano continuum that is a topological fractal contains a self regenerating fractal as a subset with nonempty interior, then every Peano continuum contains a self regenerating fractal as a subset with nonempty interior, and thus in particular every Peano continuum is a topological fractal (Hata's conjecture).

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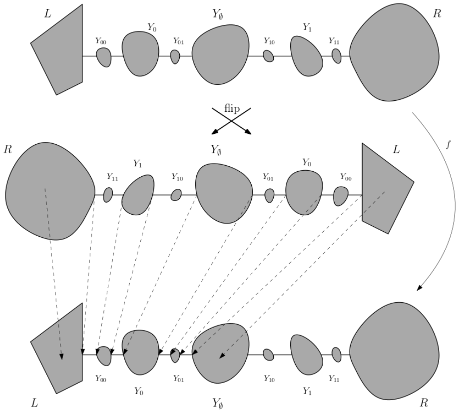
Encore



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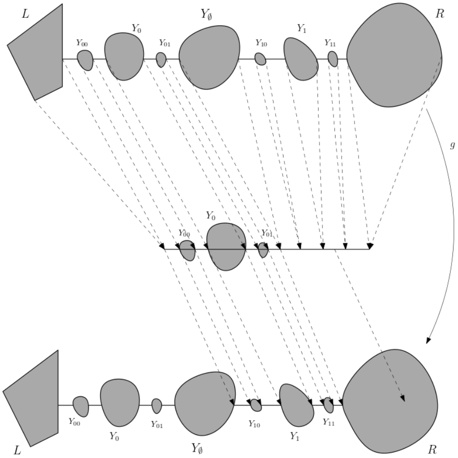

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