Forcing over a free Suslin tree

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January 28, 2024

ω_1-trees

Recall the following definitions:

Definition

We say that an ω_1-tree T is *Aronszajn* if it has no cofinal branches.

Definition

We say that an ω_1-tree T is *Suslin* if it is Aronszajn and does not contain uncountable antichains.

Definition

We say that an ω_1-tree T is *Kurepa* if it has at least ω_2-many cofinal branches.

Kurepa hypothesis

We say that the *Kurepa Hypothesis*, KH, holds if there exists a Kurepa tree.

- (Jensen) $\diamondsuit^+$ implies the existence of a Kurepa tree.
- (Silver) Silver proved the consistency of $\neg$KH by showing that after forcing with the Lévy collapse to turn an inaccessible cardinal κ into ω_2, there are no Kurepa trees.
- (Solovay) The inaccessible cardinal is necessary because if ω_2 is not an inaccessible cardinal in L then there exists a Kurepa tree.

Adding a Kurepa tree

- Jin and Shelah asked whether it is consistent that CH holds, there are no Kurepa trees, and there exists a *ccc* forcing of *size at most* ω_1 which adds a Kurepa tree.
- This question was motivated in part by Shelah's result that adding a single Cohen real adds a Suslin tree.
- The model is not a Lévy collapse of an inaccessible cardinal which becomes the new ω_2.

A result of Jensen and Schlechta

- (Jensen) If $\Box_{\omega_1}$ holds then there is a ccc forcing which adds an ω_1-Kurepa tree. However this ccc forcing has size ω_2.
- (Jensen and Schlechta) There is a model where $\neg$KH holds and it is indestructible under all ccc forcings.
- The model is the Lévy collapse of a Mahlo cardinal which becomes the new ω_2.
- Note that $\neg$KH holds already in the Lévy collapse of an inaccessible cardinal but for the indestructibility under ccc forcings a Mahlo cardinal is needed.

Result of Jin and Shelah

- Jin and Shelah proved that assuming the existence of an inaccessible cardinal κ, there exists a forcing which preserves ω_1, makes κ the new ω_2, forces that there are no Kurepa trees, and introduces a countably distributive Aronszajn tree which as a forcing notion produces a Kurepa tree.

Almost Kurepa Suslin tree

In the Jin-Shelah model, forcing with the Aronszajn tree introduces another tree which is a Kurepa tree.
Now we define a stronger concept:

Definition

We say that an ω_1-tree T is an *almost Kurepa Suslin tree* if it is a Suslin tree which becomes a Kurepa tree after forcing with it.

Note: When forcing with a tree T, the order is the reverse order of the tree T.

Existence of an almost Kurepa Suslin tree

- (Bilaniuk) $\diamondsuit$ and KH imply that there is an almost Kurepa Suslin tree.
- Bilaniuk constructed the almost Kurepa Suslin tree S equipped with at least ω_2-many automorphisms $\{\pi_b \mid b \text{ is cofinal branch } T\}$, where T is a Kurepa tree. Moreover, for each $b \neq b'$ and $s \in S$ there is $t \geq s$ such that $\pi_b(t) \neq \pi_{b'}(t)$.
- Bilaniuk asked whether it is consistent that $\diamondsuit$ holds, there are no Kurepa trees, and there exists a Suslin tree with at least ω_2-many automorphisms.
- Connected to this question is the question whether it is consistent that there exists an almost Kurepa Suslin tree and there are no Kurepa trees.

Subtrees

For the purposes of this talk we say that S is a subtree of an ω_1-tree T if it is closed downward under $\leq_T$ and it is an ω_1-tree.

Definition

We say that an Aronszajn tree T is *saturated* if every almost disjoint family of subtrees of T has cardinality at most ω_1.

Product of trees

Definition

Let S and T be trees of height ω_1. Let $S \otimes T$ denote the set of all pairs (s, t) such that there is an ordinal $\gamma < \omega_1$, $s \in S_\gamma$ and $t \in T_\gamma$. The ordering of $S \otimes T$ is component-wise: $(s, t) <_{S \otimes T} (s', t')$ if $s <_S s'$ and $t <_T t'$.

The definition can be easily generalized to products of finitely many trees.

Saturation of trees

Let S and T be ω_1-trees:

- If S is Kurepa and T is Aronszajn then $S \otimes T$ is non-saturated. Consider $U_b = \{(x, y) \mid x \in b \text{ and } y \in T\}$ for a cofinal branch b in S.
- (Baumgartner) There is a model where every Aronszajn tree is saturated. The model is given by Lévy collapse $\mathrm{Coll}(\omega_1, < \kappa)$ where κ is an inaccessible cardinal.
- Moore asked whether it is consistent that there exists a non-saturated Aronszajn tree and there are no Kurepa trees.

List of the questions which we solved (all affirmatively):

1. (Jin and Shelah, 1997) Is it consistent that CH holds, there are no Kurepa trees, and there exists a ccc forcing of size at most ω_1 which adds a Kurepa tree?
2. Is it consistent that there exists an almost Kurepa Suslin tree and there are no Kurepa trees?
3. (Bilaniuk, 1989) Is it consistent that $\diamondsuit$ holds, there are no Kurepa trees, and there exists a Suslin tree with at least ω_2-many automorphisms?
4. (Moore, 2008) Is it consistent that there exists a non-saturated Aronszajn tree and there are no Kurepa trees?

In the generic extension by the Lévy collapse which turns an inaccessible cardinal κ into ω_2 the following hold:

- (Silver) There are no Kurepa trees.
- (Folklore) There is no ccc forcing of size at most ω_1 which adds an Kurepa tree; consequently there are no almost Kurepa Suslin trees.
- (Bilaniuk) Every Suslin tree has at most ω_1-many automorphisms.
- (Baumgartner) Every Aronszajn tree is saturated.

Free Suslin tree

Definition

Let T be an ω_1-tree and let $0 < n < \omega$. A *derived tree of dimension n* (or *n-derived tree*) is a tree of the form

$$T_{t_0} \otimes T_{t_1} \otimes \cdots \otimes T_{t_{n-1}}, \tag{1}$$

where $t_0, \ldots, t_{n-1}$ are distinct elements of T of the same height.

A derived tree of dimension 1 is just a tree of the form T_t where $t \in T$.

Definition

Let $1 \leq n < \omega$. A Suslin tree T is *n-free* if all of its n-derived trees are Suslin. A Suslin tree T is *free* if it is n-free for all $1 \leq n < \omega$.

Our result

Theorem

Suppose that there exists an inaccessible cardinal κ and there exists an infinitely splitting normal free Suslin tree S. Then there exists a forcing poset $\mathbb{P}$ such that the product $\mathrm{Coll}(\omega_1, < \kappa) \times \mathbb{P}$ forces:

- $\kappa = \omega_2$;
- GCH;
- *S is Suslin;*
- *there exist an almost disjoint family $\{f_\alpha \mid \alpha < \omega_2\}$ of automorphisms of S;*
- *there are no Kurepa trees.*

Solution of the Jin and Shelah's question

It is consistent that CH holds, there are no Kurepa trees, and there exists a ccc forcing of size at most ω_1 which adds a Kurepa tree.

- If b is a generic branch obtained by forcing with the Suslin tree S over a generic extension by $\mathrm{Coll}(\omega_1, < \kappa) \times \mathbb{P}$, then $\{f_\alpha[b] \mid \alpha < \omega_2\}$ is a family of ω_2-many cofinal branches of S.
- Thus, in the generic extension by $\mathrm{Coll}(\omega_1, < \kappa) \times \mathbb{P}$, the Suslin tree S is an almost Kurepa Suslin tree. In particular it is a ccc forcing of size ω_1 which forces the existence of a Kurepa tree.
- Starting with a model with an inaccessible cardinal and forcing the existence of an infinitely splitting normal free Suslin tree (for example, by Jech's forcing), we get the following corollary which solves questions (1) and (2).

Solution of the Bilaniuk's question

It is consistent that $\diamondsuit$ holds, there are no Kurepa trees, and there exists a Suslin tree with at least ω_2-many automorphisms.

- It suffices to find a generic extension as in our theorem above which satisfies $\diamondsuit$.
- Start with a model V in which there exists an inaccessible cardinal κ and $\diamondsuit$ holds.
- Let $\mathbb{Q}$ be the Jech's forcing in V for adding a Suslin tree. Let $\dot{\mathbb{P}}$ be a $\mathbb{Q}$-name for the forcing described in the theorem above using the generic Suslin tree.

Solution of the Bilaniuk's question (cont'd)

- Since $\mathbb{Q}$ is ω_1-closed, the forcings $\mathbb{Q} * (\mathrm{Coll}(\omega_1, < \kappa)^{V^{\mathbb{Q}}} \times \dot{\mathbb{P}})$ and $(\mathbb{Q} * \dot{\mathbb{P}}) \times \mathrm{Coll}(\omega_1, < \kappa)$ are forcing equivalent.
- It holds that $\mathbb{Q} * \dot{\mathbb{P}}$ is forcing equivalent to an ω_1-closed forcing, and consequently so is $(\mathbb{Q} * \dot{\mathbb{P}}) \times \mathrm{Coll}(\omega_1, < \kappa)$
- But ω_1-closed forcings preserve $\diamondsuit$, so $\diamondsuit$ holds in the generic extension by $\mathbb{Q} * (\mathrm{Coll}(\omega_1, < \kappa)^{V^{\mathbb{Q}}} \times \dot{\mathbb{P}})$.

Solution of the Moore's question

It is consistent that there exists a non-saturated Aronszajn tree and there are no Kurepa trees.

- Working in the generic extension by $\mathrm{Coll}(\omega_1, < \kappa) \times \mathbb{P}$, for all $\alpha < \omega_2$ let $U_\alpha = \{(x, f_\alpha(x)) \mid x \in S\}$.
- Then each U_α is an uncountable downwards closed subtree of the Aronszajn tree $S \otimes S$, and any two such subtrees have countable intersection.
- So the family $\{U_\alpha \mid \alpha < \omega_2\}$ witnesses that $S \otimes S$ is not saturated.

Moore observed that if S is an almost Kurepa Suslin tree then $S \otimes S$ is non-saturated.

- Suppose that $\{\dot{b}_\alpha \mid \alpha < \omega_2\}$ is a sequence of S-names for distinct cofinal branches of S .
- For each $\alpha < \omega_2$ let U_α be the downward closure of the set $\{(x, y) \in S \otimes S \mid x \Vdash y \in \dot{b}_\alpha\}$.
- U_α is uncountable since $\dot{b}_\alpha$ is a name for a cofinal branch of S.
- Using that S is Suslin one can show that any two such subtrees have countable intersection.
- So the family $\{U_\alpha \mid \alpha < \omega_2\}$ witnesses that $S \otimes S$ is not saturated.

Corollaries of our result:

1. It is consistent that CH holds, there are no Kurepa trees, and there exists a ccc forcing of size at most ω_1 which adds a Kurepa tree.
2. It is consistent that there exists an almost Kurepa Suslin tree and there are no Kurepa trees.
3. It is consistent that $\diamondsuit$ holds, there are no Kurepa trees, and there exists a Suslin tree with at least ω_2-many automorphisms.
4. It is consistent that there exists a non-saturated Aronszajn tree and there are no Kurepa trees.

Open questions

1. Is it consistent that there exists a non-saturated Aronszajn tree and there are no almost Suslin Kurepa trees?
2. Is it consistent that there exists an almost Suslin Kurepa tree and every Suslin tree has at most ω_1-many automorphisms?