The SLO principle for Borel subsets of the generalized Cantor space

Beatrice Pitton

beatrice.pitton@unil.ch
Université de Lausanne and Università degli Studi di Torino

Joint work with Luca Motto Ros and Philipp Schlicht

Winter School in Abstract Analysis 2024 – section Set Theory and Topology

Generalized descriptive set theory

GDST

The higher analogue of classical descriptive set theory, obtained by replacing ω with κ.

Our setup

Let κ be an uncountable cardinal that satisfies the condition $\kappa^{<\kappa} = \kappa$.

Remark: Let κ be an infinite cardinal. Then $\kappa^{<\kappa} = \kappa$ is equivalent to $2^{<\kappa} = \kappa$ and κ *regular*.

The generalized Cantor and Baire spaces

Classical spaces

The Cantor space ${}^{\omega}2$ and the Baire space ${}^{\omega}\omega$. Let $A \in \{2, \omega\}$, we equip ${}^{\omega}A = \{f \mid f : \omega \to A\}$ with the topology generated by the sets

$$N_s({}^{\omega}A) := \{ x \in {}^{\omega}A \mid s \subseteq x \}, \qquad s \in {}^{<\omega}A.$$

The generalized Cantor and Baire spaces

Classical spaces

The Cantor space ${}^{\omega}2$ and the Baire space ${}^{\omega}\omega$. Let $A \in \{2, \omega\}$, we equip ${}^{\omega}A = \{f \mid f : \omega \to A\}$ with the topology generated by the sets

$$N_s({}^{\omega}A) := \{\, x \in {}^{\omega}A \mid s \subseteq x \,\}, \qquad s \in {}^{<\omega}A.$$

Let λ, κ be cardinals, with κ infinite and $\lambda \geq 2$.
We equip the set ${}^{\kappa}\lambda = \{x \mid x : \kappa \to \lambda\}$ with the *bounded topology* τ_b, generated by the sets

$$N_s({}^{\mu}\lambda) := \{\, x \in {}^{\kappa}\lambda \mid s \subseteq x \,\}, \qquad s \in {}^{<\kappa}\lambda.$$

The generalized Cantor and Baire spaces

Classical spaces

The Cantor space ${}^{\omega}2$ and the Baire space ${}^{\omega}\omega$. Let $A \in \{2, \omega\}$, we equip ${}^{\omega}A = \{f \mid f : \omega \to A\}$ with the topology generated by the sets

$$N_s({}^{\omega}A) := \{\, x \in {}^{\omega}A \mid s \subseteq x \,\}, \qquad s \in {}^{<\omega}A.$$

Let λ, κ be cardinals, with κ infinite and $\lambda \geq 2$.
We equip the set ${}^{\kappa}\lambda = \{x \mid x : \kappa \to \lambda\}$ with the *bounded topology* τ_b, generated by the sets

$$N_s({}^{\mu}\lambda) := \{\, x \in {}^{\kappa}\lambda \mid s \subseteq x \,\}, \qquad s \in {}^{<\kappa}\lambda.$$

- Generalized Cantor space $({}^{\kappa}2, \tau_b)$.

- Generalized Baire space $({}^{\kappa}\kappa, \tau_b)$.

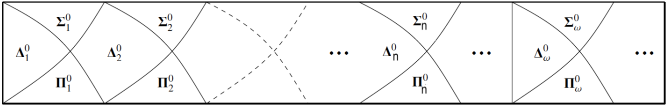
Generalized descriptive set theory

Classical definition

Let $(X, \tau) = ({}^{\omega}2, \tau)$. The Borel sets are ω_1-algebra generated by τ, i.e. the smallest collection of subsets of ${}^{\omega}2$ containing all open sets and closed under complements and unions of size $\leq \omega$.

Generalized definition

Let $(X, \tau) = ({}^{\kappa}2, \tau_b)$. The κ^+-Borel sets $\mathbf{Bor}(\kappa^+)$ are κ^+-algebra generated by τ_b, i.e. the smallest collection of subsets of ${}^{\kappa}2$ containing all open sets and closed under complements and unions of size $\leq \kappa$.



Wadge Reductions

Definition

Given $A, B \subseteq {}^{\omega}2$, let

$$A \leq_{\mathsf{W}} B$$

if there exists a continuous $f : {}^{\omega}2 \to {}^{\omega}2$ such that $f^{-1}(B) = A$.

Read: there is a continuous reduction from A to B or, A continuously reduces (or Wadge reduces) to B.

Wadge Reductions

Definition

Given $A, B \subseteq {}^{\omega}2$, let

$$A \leq_{\mathrm{W}} B$$

if there exists a continuous $f : {}^{\omega}2 \to {}^{\omega}2$ such that $f^{-1}(B) = A$.

Read: there is a continuous reduction from A to B or, A continuously reduces (or Wadge reduces) to B.

- Notice that $A \leq_{\mathrm{W}} B$ if and only if $\neg A \leq_{\mathrm{W}} \neg B$.
- Continuous reducibility is a transitive and reflexive relation, that is, a quasiorder.

Wadge Hierarchy

Definition

Given $A, B \subseteq {}^{\omega}2$, let

$$A \leq_{\mathrm{W}} B$$

if there exists a continuous $f : {}^{\omega}2 \to {}^{\omega}2$ such that $f^{-1}(B) = A$.

We set:

- $A <_{\mathrm{W}} B$ iff $A \leq_{\mathrm{W}} B$ and $B \not\leq_{\mathrm{W}} A$.
- $A \equiv_{\mathrm{W}} B$ iff $A \leq_{\mathrm{W}} B$ and $B \leq_{\mathrm{W}} A$.

The equivalence classes induced by $\leq_{\mathrm{W}}$ are called Wadge degrees

$$[A]_{\mathrm{W}} = \{B \mid A \equiv_{\mathrm{W}} B\}$$

Wadge Hierarchy

Definition

Given $A, B \subseteq {}^{\omega}2$, let

$$A \leq_{\mathrm{W}} B$$

if there exists a continuous $f : {}^{\omega}2 \to {}^{\omega}2$ such that $f^{-1}(B) = A$.

We set:

- $A <_{\mathrm{W}} B$ iff $A \leq_{\mathrm{W}} B$ and $B \not\leq_{\mathrm{W}} A$.
- $A \equiv_{\mathrm{W}} B$ iff $A \leq_{\mathrm{W}} B$ and $B \leq_{\mathrm{W}} A$.

The equivalence classes induced by $\leq_{\mathrm{W}}$ are called Wadge degrees

$$[A]_{\mathrm{W}} = \{B \mid A \equiv_{\mathrm{W}} B\}$$

Note that the quasiorder $\leq_{\mathrm{W}}$ induces a partial order on the Wadge degrees: We call this partial order the **Wadge hierarchy** on the Cantor space.

Wadge game

For any $A, B \subseteq {}^{\omega}2$, the Wadge game $G_{\mathrm{W}}(A,B)$ on ${}^{\omega}2$ is:

I	x_0		x_1		x_2		$\cdots$	
II		y_0		P		y_1		$\cdots$

Player **II** is allowed to "pass" at some stages.
Player **II** wins the game if $y \in {}^{\omega}2$ and $x \in A \iff y \in B$.

Fact

- **II** has a winning strategy in $G_{\mathrm{W}}(A,B) \iff A \leq_{\mathrm{W}} B$.
- **I** has a winning strategy in $G_{\mathrm{W}}(A,B) \implies {}^{\omega}2 \setminus B \leq_{\mathrm{W}} A$.

Wadge Hierarchy

The *Wadge Semi-Linear Ordering principle* ($\mathsf{SLO}^{\mathsf{W}}$) is the statement:
For all sets $A, B \subseteq {}^{\omega}2$

$$A \leq_{\mathsf{W}} B \quad \text{or} \quad {}^{\omega}2 \setminus B \leq_{\mathsf{W}} A.$$

Given $\boldsymbol{\Gamma}$ boldface pointclass, we write $\mathsf{SLO}^{\mathsf{W}}(\boldsymbol{\Gamma})$ if $\mathsf{SLO}^{\mathsf{W}}$ holds for any $A, B \in \boldsymbol{\Gamma}$.

Wadge Hierarchy

The *Wadge Semi-Linear Ordering principle* ($\mathsf{SLO}^{\mathsf{W}}$) is the statement:
For all sets $A, B \subseteq {}^{\omega}2$

$$A \leq_{\mathsf{W}} B \quad \text{or} \quad {}^{\omega}2 \setminus B \leq_{\mathsf{W}} A.$$

Given $\boldsymbol{\Gamma}$ boldface pointclass, we write $\mathsf{SLO}^{\mathsf{W}}(\boldsymbol{\Gamma})$ if $\mathsf{SLO}^{\mathsf{W}}$ holds for any $A, B \in \boldsymbol{\Gamma}$.

Wadge's Lemma

For all $A, B \in \mathbf{Bor}({}^{\omega}2)$,

$$A \leq_{\mathsf{W}} B \quad \text{or} \quad {}^{\omega}2 \setminus B \leq_{\mathsf{W}} A.$$

Wadge Hierarchy

The *Wadge Semi-Linear Ordering principle* ($\mathsf{SLO}^{\mathsf{W}}$) is the statement:
For all sets $A, B \subseteq {}^{\omega}2$

$$A \leq_{\mathsf{W}} B \quad \text{or} \quad {}^{\omega}2 \setminus B \leq_{\mathsf{W}} A.$$

Given $\mathbf{\Gamma}$ boldface pointclass, we write $\mathsf{SLO}^{\mathsf{W}}(\mathbf{\Gamma})$ if $\mathsf{SLO}^{\mathsf{W}}$ holds for any $A, B \in \mathbf{\Gamma}$.

Wadge's Lemma

For all $A, B \in \mathbf{Bor}({}^{\omega}2)$,

$$A \leq_{\mathsf{W}} B \quad \text{or} \quad {}^{\omega}2 \setminus B \leq_{\mathsf{W}} A.$$

Theorem (Martin, Monk)

$\leq_{\mathsf{W}}$ is well-founded on $\mathbf{Bor}({}^{\omega}2)$.

Some consequences of $\mathsf{SLO}^{\mathsf{W}}$

- Antichains have size at most 2, and they are of the form $\{[A]_{\mathsf{W}}, [\neg A]_{\mathsf{W}}\}$ for some $A \subseteq {}^{\omega}2$ such that $A \not\leq_{\mathsf{W}} \neg A$.

Some consequences of $\mathsf{SLO}^{\mathsf{W}}$

- Antichains have size at most 2, and they are of the form $\{[A]_{\mathsf{W}}, [\neg A]_{\mathsf{W}}\}$ for some $A \subseteq {}^{\omega}2$ such that $A \not\leq_{\mathsf{W}} \neg A$.

Definition

Let $\mathbf{\Gamma}$ be a boldface pointclass.

- A set $A \subseteq {}^{\omega}2$ is $\mathbf{\Gamma}$-*hard* if for all $B \in \mathbf{\Gamma}({}^{\omega}2)$, $B \leq_{\mathsf{W}} A$.
- The set A is $\mathbf{\Gamma}$-*complete* if it is $\mathbf{\Gamma}$-hard and $A \in \mathbf{\Gamma}({}^{\omega}2)$.

Some consequences of $\mathsf{SLO}^{\mathsf{W}}$

- Antichains have size at most 2, and they are of the form $\{[A]_{\mathsf{W}}, [\neg A]_{\mathsf{W}}\}$ for some $A \subseteq {}^{\omega}2$ such that $A \not\leq_{\mathsf{W}} \neg A$.

Definition

Let $\mathbf{\Gamma}$ be a boldface pointclass.

- A set $A \subseteq {}^{\omega}2$ is $\mathbf{\Gamma}$-*hard* if for all $B \in \mathbf{\Gamma}({}^{\omega}2)$, $B \leq_{\mathsf{W}} A$.
- The set A is $\mathbf{\Gamma}$-*complete* if it is $\mathbf{\Gamma}$-hard and $A \in \mathbf{\Gamma}({}^{\omega}2)$.

- Assume $\mathsf{SLO}^{\mathsf{W}}$ holds. Let $\mathbf{\Gamma}$ be a non selfdual boldface pointclass, then

$$A \text{ is } \mathbf{\Gamma}\text{-complete} \iff A \in \mathbf{\Gamma}({}^{\omega}2) \setminus \check{\mathbf{\Gamma}}({}^{\omega}2).$$

Some consequences of $\mathsf{SLO}^{\mathsf{W}}$

- Antichains have size at most 2, and they are of the form $\{[A]_{\mathsf{W}}, [\neg A]_{\mathsf{W}}\}$ for some $A \subseteq {}^{\omega}2$ such that $A \not\leq_{\mathsf{W}} \neg A$.

Definition

Let $\mathbf{\Gamma}$ be a boldface pointclass.

- A set $A \subseteq {}^{\omega}2$ is $\mathbf{\Gamma}$-*hard* if for all $B \in \mathbf{\Gamma}({}^{\omega}2)$, $B \leq_{\mathsf{W}} A$.
- The set A is $\mathbf{\Gamma}$-*complete* if it is $\mathbf{\Gamma}$-hard and $A \in \mathbf{\Gamma}({}^{\omega}2)$.

- Assume $\mathsf{SLO}^{\mathsf{W}}$ holds. Let $\mathbf{\Gamma}$ be a non selfdual boldface pointclass, then

$$A \text{ is } \mathbf{\Gamma}\text{-complete} \Longleftarrow A \in \mathbf{\Gamma}({}^{\omega}2) \setminus \check{\mathbf{\Gamma}}({}^{\omega}2).$$

Some consequences of $\mathsf{SLO}^{\mathsf{W}}$

- Assume $\mathsf{SLO}^{\mathsf{W}}$ holds. Let $\boldsymbol{\Gamma}$ be a non selfdual boldface pointclass, then

$$A \text{ is } \boldsymbol{\Gamma}\text{-complete} \iff A \in \boldsymbol{\Gamma}({}^{\omega}2) \setminus \check{\boldsymbol{\Gamma}}({}^{\omega}2).$$

Let $\boldsymbol{\Gamma}$ be a non selfdual boldface pointclass. If:

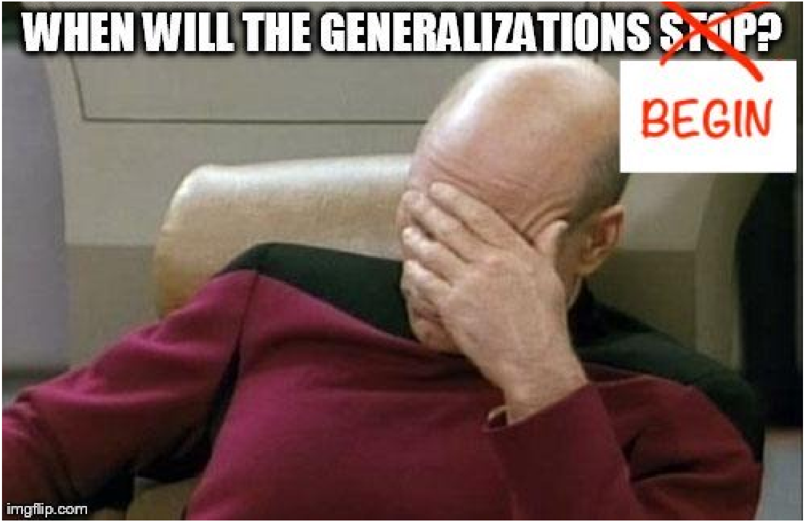
1. $\mathsf{SLO}^{\mathsf{W}}(\boldsymbol{\Gamma} \cap \check{\boldsymbol{\Gamma}})$ holds
2. A is $\boldsymbol{\Gamma}$-complete $\iff A \in \boldsymbol{\Gamma} \setminus \check{\boldsymbol{\Gamma}}$

then, $\mathsf{SLO}^{\mathsf{W}}(\boldsymbol{\Gamma})$ holds.

Some consequences of $\mathsf{SLO}^{\mathsf{W}}$

Theorem (Andretta)

$\mathsf{SLO}^{\mathsf{W}} \Longrightarrow \mathsf{PSP}$.


WHEN WILL THE GENERALIZATIONS STOP?
BEGIN
imgflip.com

Generalized Gale-Stewart game

Let κ, λ be cardinals, with κ infinite and $\lambda \geq 2$.
Given $A \subseteq {}^{\kappa}\lambda$, the generalized Gale-Stewart game $G^{\lambda}_{\kappa}(A)$ is

I	a_0		a_2		$\cdots$		a_ω		$\cdots$
II		a_1		a_3		$\cdots$		$a_{\omega+1}$	$\cdots$

Let $a := \langle a_0, a_1, \ldots, a_\omega, \ldots \rangle \in {}^{\kappa}\lambda$. Player **I** wins if $a \in A$ and **II** wins if $a \notin A$.

Generalized Gale-Stewart game

Let κ, λ be cardinals, with κ infinite and $\lambda \geq 2$.
Given $A \subseteq {}^{\kappa}\lambda$, the generalized Gale-Stewart game $G^{\lambda}_{\kappa}(A)$ is

I	a_0		a_2		$\cdots$		a_ω		$\cdots$
II		a_1		a_3		$\cdots$		$a_{\omega+1}$	$\cdots$

Let $a := \langle a_0, a_1, \ldots, a_\omega, \ldots \rangle \in {}^{\kappa}\lambda$. Player **I** wins if $a \in A$ and **II** wins if $a \notin A$.

Fact

Let $\kappa > \omega$ and let $A \subseteq {}^{\omega}2$. Then, there is an extension $\bar{A} \subseteq {}^{\kappa}2$ of A such that $\bar{A} \in \mathbf{\Delta}^0_1(\kappa^+)$ and $G^2_\kappa(\bar{A})$ is equivalent to $G^2_\omega(A)$.

$\mathsf{SLO}^{\mathsf{W}}$ in GDST

Definition

Given $A, B \subseteq {}^{\kappa}2$, let

$$A \leq_{\mathsf{W}} B$$

if there exists a continuous $f : {}^{\kappa}2 \to {}^{\kappa}2$ such that $f^{-1}(B) = A$.

The *generalized Wadge Semi-Linear Ordering principle* ($\mathsf{SLO}^{\mathsf{W}}_{\kappa}$) says:
For all sets $A, B \subseteq {}^{\kappa}2$

$$A \leq_{\mathsf{W}} B \quad \text{or} \quad {}^{\kappa}2 \setminus B \leq_{\mathsf{W}} A.$$

$\mathsf{SLO}^{\mathsf{W}}$ in GDST

Definition

Given $A, B \subseteq {}^{\kappa}2$, let

$$A \leq_{\mathsf{W}} B$$

if there exists a continuous $f : {}^{\kappa}2 \to {}^{\kappa}2$ such that $f^{-1}(B) = A$.

The *generalized Wadge Semi-Linear Ordering principle* ($\mathsf{SLO}^{\mathsf{W}}_{\kappa}$) says:
For all sets $A, B \subseteq {}^{\kappa}2$

$$A \leq_{\mathsf{W}} B \quad \text{or} \quad {}^{\kappa}2 \setminus B \leq_{\mathsf{W}} A.$$

However, there is no κ^+-Borel determinacy for $\kappa > \omega$!

$\mathsf{SLO^W}$ in GDST

Theorem (Motto Ros, P., Schlicht)

Assume $V = L$. Then, there are proper $\mathbf{\Sigma}^0_2(\kappa^+)$-sets which are not $\mathbf{\Sigma}^0_2(\kappa^+)$-complete. Hence, $\mathsf{SLO}^{\mathsf{W}}_\kappa(\mathbf{\Sigma}^0_2(\kappa^+))$ fails.

$\mathsf{SLO}^{\mathsf{W}}$ in GDST

Theorem (Lücke, Motto Ros, Schlicht)

Assume $V = L$. If κ is an uncountable regular cardinal, then there is a closed subset of ${}^{\kappa}\kappa$ that does not satisfy the Hurewicz dichotomy.

Proposition (Lücke, Motto Ros, Schlicht)

Let $\mathscr{T} \subseteq {}^{<\kappa}\kappa$ be a pruned subtree with the following three properties:

1. $\mathscr{T}$ does not contain a perfect subtree;
2. the closed set $[\mathscr{T}]$ is κ-Baire,
3. every node in $\mathscr{T}$ is κ-splitting.

Then the closed set $[\mathscr{T}]$ does not satisfy the Hurewicz dichotomy.

$\mathsf{SLO}^{\mathsf{W}}$ in GDST

Definition

Let $\mathcal{T} \subseteq {}^{<\kappa}\lambda$ with $\lambda \in \{2, \kappa\}$.

- $\mathcal{T}$ is pruned if for every $s \in \mathcal{T}$ there exists $x \in [\mathcal{T}]$ such that $s \subseteq x$.
- $\mathcal{T}$ is $<\kappa$-closed if every increasing sequence in $\mathcal{T}$ of length $<\kappa$ has an upper bound in $\mathcal{T}$,
- $\mathcal{T}$ is κ-perfect if it is $<\kappa$-closed and cofinally splitting, i.e. if for every $t \in \mathcal{T}$ there exists a splitting node $u \in \mathcal{T}$ with $t \subseteq u$.
- A subset Y of ${}^{\kappa}\lambda$ is κ-perfect if $Y = [\mathcal{T}]$ with $\mathcal{T}$ a κ-perfect tree.
- A subset A of ${}^{\kappa}\lambda$ has the κ-**perfect set property** if $|A| \leq \kappa$ or A has a κ-perfect subset.

Theorem (Motto Ros, P., Schlicht)

Assume that $\mathsf{PSP}_{\kappa}(\mathbf{\Pi}^0_1(\kappa^+))$. Then, $\mathsf{SLO}^{\mathsf{W}}_{\kappa}$ implies PSP_{κ}.

SLO^{W} in GDST

Let $\mathrm{G} = \{x \in {}^{\kappa}2 \mid \forall \alpha < \kappa \exists \beta > \alpha (x(\beta) = 0)\}$.

Theorem (Schlicht, Sziraki)

After a Levy-collapse of an inaccessible to κ^{+}, the following analogue of the *Kechris-Louveau-Woodin dichotomy* holds for all disjoint definable subsets $X, Y \subseteq {}^{\kappa}\kappa$:
Either there is a $\mathbf{\Sigma}^0_2(\kappa^+)$ set A separating X from Y, i.e. $X \subseteq A$ and $Y \cap A = \emptyset$ or there is a homeomorphism f from ${}^{\kappa}2$ onto a closed subset of ${}^{\kappa}\kappa$ such that $f(\mathrm{G}) \subseteq X$ and $f({}^{\kappa}2 \setminus \mathrm{G}) \subseteq Y$.

It is consistent that every proper $\mathbf{\Sigma}^0_2(\kappa^+)$-set is $\mathbf{\Sigma}^0_2(\kappa^+)$-complete.

SLO$^{\mathrm{W}}$ in GDST

Let $\mathrm{G} = \{x \in {}^{\kappa}2 \mid \forall \alpha < \kappa \exists \beta > \alpha (x(\beta) = 0)\}$.

Theorem (Schlicht, Sziraki)

After a Levy-collapse of an inaccessible to κ^+, the following analogue of the *Kechris-Louveau-Woodin dichotomy* holds for all disjoint definable subsets $X, Y \subseteq {}^{\kappa}\kappa$:
Either there is a $\mathbf{\Sigma}^0_2(\kappa^+)$ set A separating X from Y, i.e. $X \subseteq A$ and $Y \cap A = \emptyset$ or there is a homeomorphism f from ${}^{\kappa}2$ onto a closed subset of ${}^{\kappa}\kappa$ such that $f(\mathrm{G}) \subseteq X$ and $f({}^{\kappa}2 \setminus \mathrm{G}) \subseteq Y$.

It is consistent that every proper $\mathbf{\Sigma}^0_2(\kappa^+)$-set is $\mathbf{\Sigma}^0_2(\kappa^+)$-complete.

Question

Is it consistent that $\mathrm{SLO}^{\mathrm{W}}_{\kappa}(\mathbf{\Sigma}^0_2(\kappa^+))$ holds?

How far $\mathsf{SLO}^{\mathsf{W}}_{\kappa}$ holds

Fact 1

$\mathsf{SLO}^{\mathsf{W}}_{\kappa}(\boldsymbol{\Delta}^0_1(\kappa^+))$ holds.

How far $\mathsf{SLO}^{\mathsf{W}}_{\kappa}$ holds

Fact 1

$\mathsf{SLO}^{\mathsf{W}}_{\kappa}(\mathbf{\Delta}^0_1(\kappa^+))$ holds.

Fact 2

Let $C \subseteq {}^{\kappa}2$. If $C \in \mathbf{\Pi}^0_1(\kappa^+) \setminus \mathbf{\Sigma}^0_1(\kappa^+)$, then C is $\mathbf{\Pi}^0_1(\kappa^+)$-complete.

How far $\mathsf{SLO}^{\mathsf{W}}_{\kappa}$ holds

Fact 1

$\mathsf{SLO}^{\mathsf{W}}_{\kappa}(\mathbf{\Delta}^0_1(\kappa^+))$ holds.

Fact 2

Let $C \subseteq {}^{\kappa}2$. If $C \in \mathbf{\Pi}^0_1(\kappa^+) \setminus \mathbf{\Sigma}^0_1(\kappa^+)$, then C is $\mathbf{\Pi}^0_1(\kappa^+)$-complete.

Hence, $\mathsf{SLO}^{\mathsf{W}}_{\kappa}(\mathbf{\Sigma}^0_1(\kappa^+))$ and $\mathsf{SLO}^{\mathsf{W}}_{\kappa}(\mathbf{\Pi}^0_1(\kappa^+))$ hold.

Difference Hierarchy

$\theta \in \mathrm{Ord}$ can be uniquely written as $\theta = \lambda + n$ with λ limit or 0 and $n < \omega$.

Definition

Let $\theta \geq 1$ be an ordinal. If $(C_\eta)_{\eta<\theta}$ is a decreasing sequence of sets, we define $C = D_\theta\left((C_\eta)_{\eta<\theta}\right) \subseteq {}^\kappa 2$ by

$$x \in C \iff \begin{cases} x \in \bigcap_{\eta<\theta} C_\eta \vee \min\left\{\eta < \theta \mid x \notin C_\eta\right\} \text{ is odd} & \text{for } \theta \text{ odd} \\ x \notin \bigcap_{\eta<\theta} C_\eta \wedge \min\left\{\eta < \theta \mid x \notin C_\eta\right\} \text{ is odd} & \text{for } \theta \text{ even} \end{cases}$$

Difference Hierarchy

$\theta \in \mathrm{Ord}$ can be uniquely written as $\theta = \lambda + n$ with λ limit or 0 and $n < \omega$.

Definition

Let $\theta \geq 1$ be an ordinal. If $(C_\eta)_{\eta<\theta}$ is a decreasing sequence of sets, we define $C = D_\theta\left((C_\eta)_{\eta<\theta}\right) \subseteq {}^\kappa 2$ by

$$x \in C \iff \begin{cases} x \in \bigcap_{\eta<\theta} C_\eta \vee \min\left\{\eta < \theta \mid x \notin C_\eta\right\} \text{ is odd} & \text{for } \theta \text{ odd} \\ x \notin \bigcap_{\eta<\theta} C_\eta \wedge \min\left\{\eta < \theta \mid x \notin C_\eta\right\} \text{ is odd} & \text{for } \theta \text{ even} \end{cases}$$

For $1 \leq \theta < \kappa^+$, we let

$$\mathbf{D}_\theta\left(\mathbf{\Pi}^0_1(\kappa^+)\right) = \left\{D_\theta\left((C_\eta)_{\eta<\theta}\right) \mid C_\eta \in \mathbf{\Pi}^0_1(\kappa^+) \text{ for every } \eta < \theta\right\}.$$

We also define $\check{\mathbf{D}}_\theta\left(\mathbf{\Pi}^0_1(\kappa^+)\right)$ to be the dual class of $\mathbf{D}_\theta\left(\mathbf{\Pi}^0_1(\kappa^+)\right)$.

Difference Hierarchy - DTS

Theorem (Hausdorff, Kuratowski)

In every polish space X and for any $1 \leq \alpha < \omega_1$,

$$\mathbf{\Delta}^0_{\alpha+1}(X) = \bigcup_{1 \leq \theta < \omega_1} \mathsf{D}_\theta\left(\mathbf{\Pi}^0_\alpha(X)\right)$$

A counterexample to Hausdorff-Kuratowski in GDST

Theorem

Let $X \subseteq {}^{\kappa}2$. If $Y \subseteq X$ is non-empty, dense and codense in $\overline{X}$, then $Y \notin \mathbf{D}_{\theta}(\mathbf{\Pi}^0_1(X, \kappa^+))$ for any $\theta < \kappa^+$.

Consider the sets

$$X := \{x \in {}^{\kappa}2 \mid |\{\alpha < \kappa \mid x(\alpha) = 1\}| < \aleph_0\}$$

and

$$Y := \{x \in {}^{\kappa}2 \mid \exists n < \omega |\{\alpha < \kappa \mid x(\alpha) = 1\}| = 2n\}.$$

Define also $Y^c := X \setminus Y = \{x \in X \mid \exists n < \omega |\{\alpha < \kappa \mid x(\alpha) = 1\}| = 2n+1\}$.

How far $\mathsf{SLO}^{\mathsf{W}}_{\kappa}$ holds

Structure of our work for $\theta > 1$:

1. Show that $\mathsf{SLO}^{\mathsf{W}}_{\kappa}(\boldsymbol{\Gamma})$ holds for $\boldsymbol{\Gamma} = \mathbf{D}_{\theta}(\boldsymbol{\Pi}^0_1(\kappa^+)) \cap \check{\mathbf{D}}_{\theta}(\boldsymbol{\Pi}^0_1(\kappa^+))$.
2. Every proper $\mathbf{D}_{\theta}(\boldsymbol{\Pi}^0_1(\kappa^+))$-subset $C \subseteq {}^{\kappa}2$ is $\mathbf{D}_{\theta}(\boldsymbol{\Pi}^0_1(\kappa^+))$-complete.
3. $\mathsf{SLO}^{\mathsf{W}}_{\kappa}(\mathbf{D}_{\theta}(\boldsymbol{\Pi}^0_1(\kappa^+)))$ holds.

Thank you!