Ultrafilters generic over $\mathcal{P}(\mathbb{N})/I$

Jonathan L. Verner
(joint work with M. Hrušák)

Hejnice, January 2011

Motivation

How to construct interesting ultrafilters ?

ZFC Constructions

Theorem

- *There is a* **free** *ultrafilter. (Zermelo ?)*

ZFC Constructions

Theorem

- *There is a* **free** *ultrafilter. (Zermelo ?)*
- *There is a weak P-point ultrafilter! (Kunen)*

ZFC Constructions II.

Definition

Suppose $\mathcal{F}$ is a filter on ω. We say a family (or a matrix) $\mathcal{X} = \{X^n_{\alpha,\beta} : n < \omega, \alpha \in \kappa, \beta \in \lambda\}$ of subsets of ω is a κ **by** λ **independent linked family w.r.t.** $\mathcal{F}$ if For each α, β, n we have $A^n_{\alpha,\beta} \subseteq A^{n+1}_{\alpha,\beta}$ (i.e. the sets increase with n), and for each finite set of indeces $L \in [\lambda]^{<\omega}$, for each function $n : L \to \omega$ and $A \in \prod_{\beta \in L} [\kappa]^{n(\beta)}$ and each $F \in \mathcal{F}$ the intersection

$$F \cap \bigcap_{\beta \in L} \bigcap_{\alpha \in A(\beta)} X^{n(\beta)}_{\alpha,\beta}$$

is infinite, while for each $\beta \in \lambda, n < \omega, A \in [\kappa]^{n+1}$ the intersection

$$\bigcap_{\alpha \in A} X^n_{\alpha,\beta}$$

is finite.

ZFC Constructions III.

Theorem (Shelah)

It is consistent with ZFC that there are no P-point ultrafilters!

CH & MA & friends

Theorem

Assuming $\mathfrak{d} = \mathfrak{c}$ there is a P-point.

CH & MA & friends

Theorem

Assuming $\mathfrak{d} = \mathfrak{c}$ there is a P-point.

- Usually inductive constructions.

CH & MA & friends

Theorem

Assuming $\mathfrak{d} = \mathfrak{c}$ there is a P-point.

- Usually inductive constructions.
- Almost all ultrafilters you can come up with.

CH & MA & friends

Theorem

Assuming $\mathfrak{d} = \mathfrak{c}$ there is a P-point.

- Usually inductive constructions.
- Almost all ultrafilters you can come up with.
- Still too much work.

Forcing (the „easy“ way)

- Force with F_σ-ideals (adds a Canjar ultrafilter).

Forcing (the „easy“ way)

- Force with F_σ-ideals (adds a Canjar ultrafilter).
- Force with analytic ideals (adds non-rapid non-p-points).

Forcing (the „easy“ way)

- Force with F_σ-ideals (adds a Canjar ultrafilter).
- Force with analytic ideals (adds non-rapid non-p-points).
- Force with $\mathcal{P}(\omega)/fin$ (adds a selective ultrafilter).

Forcing (the „easy“ way)

- Force with F_σ-ideals (adds a Canjar ultrafilter).
- Force with analytic ideals (adds non-rapid non-p-points).
- Force with $\mathcal{P}(\omega)/fin$ (adds a selective ultrafilter).

P-points

Definition

$\mathcal{U}$ is a P-point if for any descending $\langle A_n : n < \omega \rangle$ sequence of sets from $\mathcal{U}$ there is an interval partition $\langle I_n : n < \omega \rangle$ such that

$$\bigcup_{n<\omega} A_n \cap I_n \in \mathcal{U}$$

P-points

Definition

$\mathcal{U}$ is a P-point if for any descending $\langle A_n : n < \omega \rangle$ sequence of sets from $\mathcal{U}$ there is an interval partition $\langle I_n : n < \omega \rangle$ such that

$$\bigcup_{n<\omega} A_n \cap I_n \in \mathcal{U}$$

Theorem (Zapletal)

$\mathcal{U}$ is a P-point if any analytic ideal disjoint from $\mathcal{U}$ can be separated from $\mathcal{U}$ by an F_σ-ideal.

P-points, II.

Folklore

If I is an F_σ ideal then forcing with $\mathcal{P}(\omega)/I$ adds a P-point.

P-points, II.

Folklore

If I is an F_σ ideal then forcing with $\mathcal{P}(\omega)/I$ adds a P-point.

Proof.

- The key is to prove that $\mathcal{P}(\omega)/I$ is σ-closed.

P-points, II.

Folklore

If I is an F_σ ideal then forcing with $\mathcal{P}(\omega)/I$ adds a P-point.

Proof.

- The key is to prove that $\mathcal{P}(\omega)/I$ is σ-closed.
- Use Mazur's theorem, that $I = Fin(\mu) = \{X \subseteq \omega : \mu(X) < \infty\}$ for some lscsm μ.

□

P-points, III.

Theorem

$\mathcal{P}(\omega)/I$ adds a P-point iff I is locally F_σ.

P-points, III.

Theorem

$\mathcal{P}(\omega)/I$ adds a P-point iff I is locally F_σ.

Proof.

- Choose a condition A and a generic G containing the condition A.

P-points, III.

Theorem

$\mathcal{P}(\omega)/I$ adds a P-point iff I is locally F_σ.

Proof.

- Choose a condition A and a generic G containing the condition A.
- Use Zapletal's theorem in the extension to find an F_σ-ideal J disjoint from the generic.

P-points, III.

Theorem

$\mathcal{P}(\omega)/I$ adds a P-point iff I is locally F_σ.

Proof.

- Choose a condition A and a generic G containing the condition A.
- Use Zapletal's theorem in the extension to find an F_σ-ideal J disjoint from the generic.
- Since the forcing is σ-closed, J is in the groundmodel.

P-points, III.

Theorem

$\mathcal{P}(\omega)/I$ adds a P-point iff I is locally F_σ.

Proof.

- Choose a condition A and a generic G containing the condition A.
- Use Zapletal's theorem in the extension to find an F_σ-ideal J disjoint from the generic.
- Since the forcing is σ-closed, J is in the groundmodel.
- Argue that $I \restriction A = J \restriction A$.

□

Complex, locally F_σ example

Example

There are tall Borel ideals of arbitrary high complexity which are locally F_σ.

Complex, locally F_σ example

Example

There are tall Borel ideals of arbitrary high complexity which are locally F_σ.

Proof.

Given an $A \subseteq 2^\omega$, let I_A be the ideal on $\omega^{<\omega}$ generated by

Complex, locally F_σ example

Example

There are tall Borel ideals of arbitrary high complexity which are locally F_σ.

Proof.

Given an $A \subseteq 2^\omega$, let I_A be the ideal on $\omega^{<\omega}$ generated by

- branches in A (i.e. sets of the form $\{f \restriction n : n < \omega\}$ for $f \in A$)

Complex, locally F_σ example

Example

There are tall Borel ideals of arbitrary high complexity which are locally F_σ.

Proof.

Given an $A \subseteq 2^\omega$, let I_A be the ideal on $\omega^{<\omega}$ generated by

- branches in A (i.e. sets of the form $\{f \restriction n : n < \omega\}$ for $f \in A$)
- antichains in $\omega^{<\omega}$

Complex, locally F_σ example

Example

There are tall Borel ideals of arbitrary high complexity which are locally F_σ.

Proof.

Given an $A \subseteq 2^\omega$, let I_A be the ideal on $\omega^{<\omega}$ generated by

- branches in A (i.e. sets of the form $\{f \restriction n : n < \omega\}$ for $f \in A$)
- antichains in $\omega^{<\omega}$
- sets of the form $\{f \restriction n : n \in X\}$ for $f \notin A$, $X \in I_{1/n}$

□

Selective ultrafilters

Definition

$\mathcal{U}$ is selective if for any partition of ω into sets from $\mathcal{U}$ there is a selector in $\mathcal{U}$.

Selective ultrafilters

Definition

$\mathcal{U}$ is selective if for any partition of ω into sets from $\mathcal{U}$ there is a selector in $\mathcal{U}$.

Theorem (Mathias)

$\mathcal{U}$ is selective iff it is disjoint from each tall analytic ideal.

Selective ultrafilters

Definition

$\mathcal{U}$ is selective if for any partition of ω into sets from $\mathcal{U}$ there is a selector in $\mathcal{U}$.

Theorem (Mathias)

$\mathcal{U}$ is selective iff it is disjoint from each tall analytic ideal.

Corollary

$\mathcal{P}(\omega)/I$ adds a selective ultrafilter iff I is locally fin.

Katětov-Blass order, $\mathcal{ED}_{fin}$

Definition

An ideal I is Katětov-Blass below J ($I \leq_{KB} J$) if there is a finite-to-one $f : \omega \to \omega$ such that preimages of I-small sets are J-small.

Katětov-Blass order, $\mathcal{E}D_{fin}$

Definition

An ideal I is Katětov-Blass below J ($I \leq_{KB} J$) if there is a finite-to-one $f : \omega \to \omega$ such that preimages of I-small sets are J-small.

Definition

$\mathcal{E}D_{fin}$ is the ideal on $\{(x, y) : x \leq y, x, y \in \omega\}$ consisting of sets which can be covered by finitely many functions

Katětov-Blass order, $\mathcal{E}D_{fin}$

Definition

An ideal I is Katětov-Blass below J ($I \leq_{KB} J$) if there is a finite-to-one $f : \omega \to \omega$ such that preimages of I-small sets are J-small.

Definition

$\mathcal{E}D_{fin}$ is the ideal on $\{(x, y) : x \leq y, x, y \in \omega\}$ consisting of sets which can be covered by finitely many functions

Definition

An ideal I is summable, if there is some $g : \omega \to \mathbb{R}_0^+$ such that

$$I = \left\{ X \subseteq \omega : \sum_{n \in X} f(n) < \infty \right\}$$

Q-points

Definition

An ultrafilter $\mathcal{U}$ is a Q-point if for any interval partition $\langle I_n : n < \omega \rangle$ of ω there is a selector in $\mathcal{U}$.

Q-points

Definition

An ultrafilter $\mathcal{U}$ is a Q-point if for any interval partition $\langle I_n : n < \omega \rangle$ of ω there is a selector in $\mathcal{U}$.

Theorem

$\mathcal{P}(\omega)/I$ adds a Q-point iff I is locally not KB-above fin.

Rapid ultrafilters

Definition

An ultrafilter $\mathcal{U}$ is rapid if for any interval partition $\langle I_n : n < \omega \rangle$ of ω there is an $A \in \mathcal{U}$ such that $|A \cap I_n| \leq n$.

Rapid ultrafilters

Definition

An ultrafilter $\mathcal{U}$ is rapid if for any interval partition $\langle I_n : n < \omega \rangle$ of ω there is an $A \in \mathcal{U}$ such that $|A \cap I_n| \leq n$.

Theorem (Vojtáš)

An ultrafilter $\mathcal{U}$ is rapid iff it intersects each tall summable ideal.

Rapid ultrafilters

Definition

An ultrafilter $\mathcal{U}$ is rapid if for any interval partition $\langle I_n : n < \omega \rangle$ of ω there is an $A \in \mathcal{U}$ such that $|A \cap I_n| \leq n$.

Theorem (Vojtáš)

An ultrafilter $\mathcal{U}$ is rapid iff it intersects each tall summable ideal.

Theorem

$\mathcal{P}(\omega)/I$ adds a rapid ultrafilter iff I is locally not KB-above a tall summable ideal.

Canjar ultrafilters

Definition

An ultrafilter is Canjar if forcing with $\mathbb{M}_{\mathcal{U}}$ does not add a dominating real.

Canjar ultrafilters

Definition

An ultrafilter is Canjar if forcing with $\mathbb{M}_{\mathcal{U}}$ does not add a dominating real.

Conjecture (Laflamme)

An ultrafilter is Canjar iff it is a P-point with no rapid RK-predecessor

Canjar ultrafilters

Definition

An ultrafilter is Canjar if forcing with $\mathbb{M}_{\mathcal{U}}$ does not add a dominating real.

Conjecture (Laflamme)

An ultrafilter is Canjar iff it is a P-point with no rapid RK-predecessor

Theorem

If I is a tall F_σ P-ideal then $\mathcal{P}(\omega)/I$ does adds a P-point with no rapid RK-predecessors which is **not** *Canjar*

HM-condition, strong P-points

Theorem (Hrušák-Minami)

$\mathcal{U}$ is Canjar iff each descending sequence $X_n \in ([\mathcal{U}]^{<\omega})^+$ has a pseudointersection in $([\mathcal{U}]^{<\omega})^+$, where

HM-condition, strong P-points

Theorem (Hrušák-Minami)

$\mathcal{U}$ is Canjar iff each descending sequence $X_n \in ([\mathcal{U}]^{<\omega})^+$ has a pseudointersection in $([\mathcal{U}]^{<\omega})^+$, where

$$([\mathcal{U}]^{<\omega})^+ = \{X \subseteq [\omega]^{<\omega} : (\forall F \in \mathcal{U})(\exists a \in X)(a \subseteq F)\}$$

HM-condition, strong P-points

Theorem (Hrušák-Minami)

$\mathcal{U}$ is Canjar iff each descending sequence $X_n \in ([\mathcal{U}]^{<\omega})^+$ has a pseudointersection in $([\mathcal{U}]^{<\omega})^+$, where

$$([\mathcal{U}]^{<\omega})^+ = \{X \subseteq [\omega]^{<\omega} : (\forall F \in \mathcal{U})(\exists a \in X)(a \subseteq F)\}$$

Theorem (Blass, Laflamme)

$\mathcal{U}$ is Canjar iff it is a strong P-point.

Disproof of Laflamme's conjecture

Proof.

- For the first part use Vojtáš's characterization of rapid ultrafilters and Mazur's theorem.

Disproof of Laflamme's conjecture

Proof.

- For the first part use Vojtáš's characterization of rapid ultrafilters and Mazur's theorem.
- Compute a little bit.

Disproof of Laflamme's conjecture

Proof.

- For the first part use Vojtáš's characterization of rapid ultrafilters and Mazur's theorem.
- Compute a little bit.
- Use characterization of Canjar ultrafilters due to Hrušák and Minami.

Disproof of Laflamme's conjecture

Proof.

- For the first part use Vojtáš's characterization of rapid ultrafilters and Mazur's theorem.
- Compute a little bit.
- Use characterization of Canjar ultrafilters due to Hrušák and Minami.
- Let $I = Fin(\mu) = Exh(\mu)$, $X_n = \{a \in [\omega]^{<\omega} : \mu(a) \geq n\}$.

Disproof of Laflamme's conjecture

Proof.

- For the first part use Vojtáš's characterization of rapid ultrafilters and Mazur's theorem.
- Compute a little bit.
- Use characterization of Canjar ultrafilters due to Hrušák and Minami.
- Let $I = Fin(\mu) = Exh(\mu)$, $X_n = \{a \in [\omega]^{<\omega} : \mu(a) \geq n\}$.
- Compute a bit more ...

Disproof of Laflamme's conjecture

Proof.

- For the first part use Vojtáš's characterization of rapid ultrafilters and Mazur's theorem.
- Compute a little bit.
- Use characterization of Canjar ultrafilters due to Hrušák and Minami.
- Let $I = Fin(\mu) = Exh(\mu)$, $X_n = \{a \in [\omega]^{<\omega} : \mu(a) \geq n\}$.
- Compute a bit more ...
- and you are done.

□

Existence of Canjar ultrafilters

Theorem (Canjar)

Forcing with F_σ-ideal adds a Canjar ultrafilter.

Existence of Canjar ultrafilters

Theorem (Canjar)

Forcing with F_σ-ideal adds a Canjar ultrafilter.

Question

Is there an ideal I such that $\mathcal{P}(\omega)/I$ adds a Canjar ultrafilter?

Examples

Example

$\mathcal{P}(\omega)/\mathcal{ED}_{fin}$ adds a rapid with a selective RB-below.

Examples

Example

$\mathcal{P}(\omega)/\mathcal{E}D_{fin}$ adds a rapid with a selective RB-below.

Example

$fin \times fin$ adds a Q-point which is not a P-point.

Examples

Example

$\mathcal{P}(\omega)/\mathcal{ED}_{fin}$ adds a rapid with a selective RB-below.

Example

$fin \times fin$ adds a Q-point which is not a P-point.

Example

$\mathcal{P}(\omega)/\mathcal{G}_C$ adds a rapid which is neither a P-point nor a Q-point.