On Generic Independent Families

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Introduction: Independence

This is joint work with Vera Fischer.

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- An independent family is maximal if it is maximal with respect to inclusion as an independent family.
- Denote by $\mathfrak{i}$ the independence number, the least size of a maximal independent family.

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The central issue is that $\mathfrak{i}$ has many known lower bounds, notably $\mathrm{cof}(\mathcal{M})$ (Shelah-Hrušák), but no known upper bounds. Thus building a model where $\mathfrak{i}$ is small requires preserving several other cardinals small but cannot be easily accomplished.

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- Fix an independent family $\mathcal{I}$. Denote by $\mathsf{FF}(\mathcal{I})$ the set of finite partial functions $h : \mathcal{I} \to 2$. Now, given $h \in \mathsf{FF}(\mathcal{I})$ let $\mathcal{I}^h := \bigcap_{A \in \mathrm{dom}(h)} A^{h(A)}$.

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Note that being independent is equivalent to saying all such $\mathcal{I}^h$ are infinite. We refer to sets of the form $\mathcal{I}^h$ as *Boolean combinations*.

Dense Maximality

Observe that if $\mathcal{I}$ is an independent family, then it is maximal if for each $A \in [\omega]^\omega \setminus \mathcal{I}$ there is some $h \in \mathrm{FF}(\mathcal{I})$ which witnesses why $A \notin \mathcal{I}$ i.e. h is so that either $\mathcal{I}^h \subseteq^* A$ or $\mathcal{I}^h \cap A =^* \emptyset$. This observation motivates the following definition.

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Definition (Dense Maximality)

An independent family is densely maximal if for each $A \in [\omega]^\omega$ and each $h \in \mathrm{FF}(\mathcal{I})$ there is an $h' \supseteq h$ so that either $\mathcal{I}^{h'} \subseteq^* A$ or $\mathcal{I}^{h'} \cap A =^* \emptyset$.

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Informally a family $\mathcal{I}$ is densely maximal if the set of witnesses for each $A \notin \mathcal{I}$ is dense in $\mathrm{FF}(\mathcal{I})$.

Dense Maximality

We will not need it but it's worth noting that every maximal independent family is "almost" densely so. This observation first appeared in this form in a paper of Goldstern and Shelah but is perhaps more accurately due to El'kin who observed the same thing in the setting of irresolvable spaces.

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Fact

If $\mathcal{I}$ is a maximal independent family then there is a Boolean combination $\mathcal{I}^h$ so that $\mathcal{I} \upharpoonright h := \{A \cap \mathcal{I}^h \mid A \in \mathcal{I} \setminus \mathrm{dom}(h)\}$ is densely maximal as an independent family on $\mathcal{I}^h$.

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One checks easily the following proposition.

Proposition

Let $\mathcal{I}$ be an independent family. Then $\mathcal{I}$ is densely maximal if and only if $\mathrm{fil}(\mathcal{I})$ is the unique filter which is maximal with respect to the property of consisting only of $X \in [\omega]^\omega$ which have infinite intersection with every Boolean combination.

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- A Q-filter if given every partition of ω into finite sets $\{I_n \mid n < \omega\}$ there is a *semiselector* $A \in \mathcal{F}$, i.e. a set $A \in \mathcal{F}$ such that $|A \cap I_n| \leq 1$ for all $n < \omega$,

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- Ramsey (or selective) if it is both a P-filter and a Q-filter.

Call a densely maximal independent family $\mathcal{I}$ *selective* if $\mathrm{fil}(\mathcal{I})$ is selective.

Selective Independent Families

An independent family is called *selective* if it is densely maximal and $\mathrm{fil}(\mathcal{I})$ is selective. Shelah showed that under CH such families exist. Moreover these independent families are indestructible by many different forcing notions.

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Theorem

Let $\mathcal{I}$ be a selective independent family. Then $\mathcal{I}$ is remains selective (and hence maximal) after forcing with a countable support iteration of any of the following:

- *Sacks forcing (Shelah);*
- *Miller partition forcing (Cruz-Chapital-Fischer-Guzmán-Šupina)*
- *h-Perfect Tree Forcing Notions for different functions $h : \omega \to \omega$ with $1 < h(n) < \omega$ for all $N < \omega$ (S.);*
- *Miller lite forcing (Fischer-S.);*

and many more...

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Theorem

(CH) Let δ be an ordinal, $\mathcal{I}$ a selective independent family and let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \delta \rangle$ be a countable support iteration of forcing notions so that for each $\alpha < \delta$ we have $\Vdash_\alpha$ "$\dot{\mathbb{Q}}_\alpha$ is proper and Cohen preserving". If for every $\alpha < \delta$,

$$\Vdash_\alpha \text{"}\dot{\mathbb{Q}}_\alpha \textit{ preserves the dense maximality of } \mathcal{I}\text{"},$$

then $\mathbb{P}_\delta$ preserves that $\mathcal{I}$ is selective and in particular maximal.

This theorem appears implicitly in Shelah's proof of the consistency of $\mathfrak{i} < \mathfrak{u}$ and again implicitly in a paper Chodounský-Fischer-Grebik. This explicit form is in Fischer-Switzer, [3].

Adding Selective Independent Families

There is an important trend in the literature that "maximal families added by forcing" are "forcing indestructible". For instance Hechler-generic MAD families are Cohen (and random) indestructible. One expects the same to true for independent families and the goal of this part of the talk is to sketch a proof that "the usual way" to add a maximal independent family actually adds a selective one. Towards this we need to say what the "usual way" to add such a family is.

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Definition

Let $\mathcal{I}$ be an independent family. A *diagonalization filter* for $\mathcal{I}$ is a filter consisting of $X \in [\omega]^\omega$ which have infinite intersection with every Boolean combination of $\mathcal{I}$ and is, moreover, maximal with this property.

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For instance, if $\mathcal{I}$ is densely maximal then $\text{fil}(\mathcal{I})$ is the unique such filter, while in many cases there are several and no canonical way to choose one.

Adding Selective Independent Families

The point of a diagonalization filter is the following lemma due to Brendle. Recall that given a filter $\mathcal{F}$ the *Mathias forcing relative to* $\mathcal{F}$ is the forcing notion consisting of pairs (p, A) with $p \in \omega^{<\omega}$, $A \in \mathcal{F}$, $\max(\text{range}(p)) < \min(A)$ and $(p, A) \leq (q, B)$ if and only if $p \supseteq q$, $A \subseteq B$ and $\text{range}(p) \setminus \text{range}(q) \subseteq B$.

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Lemma

Let $\mathcal{I}$ be independent and $\mathcal{F}$ be a diagonalization filter. Forcing with $\mathbb{M}(\mathcal{F})$ adds a real $\dot{M}$ so that $\mathcal{I} \cup \{\dot{M}\}$ is forced to be independent and for each ground model $X \in [\omega]^\omega \setminus \mathcal{I}$ the set $\mathcal{I} \cup \{\dot{M}\} \cup \{X\}$ is forced not to be maximal.

Given this we can build a maximal independent family of any desired (uncountably cofinal) size simply by a finite support iteration of such forcing notions where, at stage α we add the generic real described above to the independent family we are building.

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Thus, modulo some choices, the generic independent family you get from the Mathias iteration is in fact selective. This has some important corollaries.

Theorem (Fischer-S.)

There may be selective independent families of any desired (uncountably cofinal) size. Moreover if $\mathfrak{p} = \mathfrak{c}$ then there are selective independent families.

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• So suppose X is forced at some stage α to have infinite intersection with every Boolean combination. If there is a stage $\gamma > \alpha$ so that X is not in the diagonalization filter chosen at stage γ then there is a Boolean combination g so that $\mathcal{I}^g \cap M_\gamma$ is almost disjoint from X contradicting the defining property of X.

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• So suppose X is forced at some stage α to have infinite intersection with every Boolean combination. If there is a stage $\gamma > \alpha$ so that X is not in the diagonalization filter chosen at stage γ then there is a Boolean combination g so that $\mathcal{I}^g \cap M_\gamma$ is almost disjoint from X contradicting the defining property of X.

• Hence X is in every diagonalization filter chosen after stage α. But then there is a tail of generic reals which are almost contained in X which suffices to see that X is in the density filter.

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Question

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For that matter we do not know the answer to the following.

Question

If there is a selective filter is there a selective independent family? What about just a Sacks indestructible one?

Thank You!

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