On the Binary Linear Ordering

Thilo Weinert
Mathematical Institute
Università degli Studi di Udine
Udine, Friuli
Italy

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Definition

An ordinal is α called *additively decomposable* if there are ordinals $\beta, \gamma < \alpha$ such that $\beta + \gamma = \alpha$, otherwise it is called *indecomposable*.

Definition

An ordinal is α called *multiplicatively decomposable* if there are ordinals $\beta, \gamma < \alpha$ such that $\beta \cdot \gamma = \alpha$, otherwise it is called *indecomposable*.

Folklore

The infinite additively indecomposable ordinals are exactly those of the form ω^{α} for some positive ordinal α.

Folklore

The infinite multiplicatively indecomposable ordinals are exactly those of the form $\omega^{\omega^{\alpha}}$ for some positive ordinal α.

Question

Which ordinals are multiplicatively indecomposable but additively decomposable?

Memory

There are none.

- 0 is additively indecomposable,
- 1 is additively indecomposable,
- 2 is additively decomposable but multiplicatively indecomposable.

Question

Which ordinals are multiplicatively indecomposable but additively decomposable?

Memory

There are no ne .

- 0 is additively indecomposable,
- 1 is additively indecomposable,
- 2 is additively decomposable but multiplicatively indecomposable.

Question

Which ordinals are multiplicatively indecomposable but additively decomposable?

Memory

There are no infinite ones.

- 0 is additively indecomposable,
- 1 is additively indecomposable,
- 2 is additively decomposable but multiplicatively indecomposable.

Definition

We call an order-type φ *transcendable* if there are types $\rho, \psi < \varphi$ such that $\varphi \leqslant \rho\psi$, otherwise we call it *untranscendable*.

Observation

- *Both 2 and 3 are additively decomposable and multiplicatively indecomposable but 3 is transcendable while 2 is untranscendable.*
- *2 is the only finite linear ordering which is additively decomposable yet untranscendable.*

Definition

We write $\rho \longrightarrow (\varphi)^m_n$ to mean that for every partition of the m-element subsets of an ordered X set of type ρ into n classes, there is a $Y \subset X$ such that all m-element subsets of Y belong to the same class, otherwise we write $\rho \not\longrightarrow (\varphi)^m_n$.

We are mostly interested in the case where $m = 1$, $n = 2$, and where $\rho = \varphi$, i.e. forms of the pigeonhole principle. If $\varphi \longrightarrow (\varphi)^1_2$ we also say that φ has Big Ramsey degree 1 for points or simply, that φ is *indivisible*.

Observation

2 *is the only divisible untranscendable ordinal* α*.*

Definition

For any linear order types ρ and φ, we let $\rho + \varphi$ be the type of the ordering $(P \cup R, <)$ where $(P, <_P)$ is an ordering of type φ and $(R, <_R)$ is an ordering of type ρ and we let

$$x < y \text{ iff } (x \in P \wedge y \in R) \tag{1}$$
$$\vee(\{x, y\} \subset P \wedge x <_P y) \tag{2}$$
$$\vee(\{x, y\} \subset R \wedge x <_R y). \tag{3}$$

Definition

For any linear order types ρ and φ, we lemt $\rho \cdot \varphi$ be the type of the ordering $(P \times R, <)$ where $(P, <_P)$ is an ordering of type φ and $(R, <_R)$ is an ordering of type ρ and we let

$$(a, x) < (b, y) \text{ iff } (a <_R b) \tag{4}$$
$$\vee(a = b \wedge x <_R y). \tag{5}$$

Notation

If $<$ is an ordering, we denote its reversal by $<^*$. That is, $a <^* b$ if and only if $b < a$.

Observation

Whereas for ordinals α and β, the statements $\alpha \simeq \beta$ and $\alpha = \beta$ are equivalent, the same fails to hold for general linear orderings. For example

$$(\omega^* + \omega)\omega \simeq (\omega^* + \omega + 1)\omega, \text{ but} \quad (6)$$

$$(\omega^* + \omega)\omega \neq (\omega^* + \omega + 1)\omega. \quad (7)$$

Definition

An *order type* is an equivalence class of orderings with respect to order-preserving bijections.

Notation

For order types φ and ρ we write $\varphi \leqslant \rho$ to say that for every ordering P of type φ and every ordering R of type ρ there is an order-preserving injection from P into R. We write

$$\varphi < \rho \text{ iff } \varphi \leqslant \rho \text{ but not } \rho \leqslant \varphi, \tag{8}$$

$$\text{and } \varphi \simeq \rho \text{ iff both } \varphi \leqslant \rho \text{ and } \rho \leqslant \varphi. \tag{9}$$

Observation

If an order-type φ is additively decomposable, then φ is divisible; the converse does not hold, consider $(\omega^* + \omega)\omega$.

Exercise

The finite linear order type of cardinality 2 is the only linear order type which is additively decomposable yet untranscendable.

Notation

$\eta = \text{otp}(\mathbb{Q})$ and $\lambda = \text{otp}(\mathbb{R})$.

Exercise

λ is untranscendable.

Theorem (Sierpiński [1932])

λ is divisible.

Definition

A linear ordering φ is called *scattered* if $\eta \not\leq \varphi$.

Theorem (Hausdorff [1908])

The class of scattered linear orderings is the closure of $\{0, 1\}$ under ordinal sums and reverse ordinal sums.

Theorem (Laver [1973])

Every scattered linear ordering is a finite sum of additively indecomposable order types.

Definition

A regular unbounded sum of order types φ_α is a sum $\sum_{\alpha<\kappa} \varphi_\alpha$ or a sum $\sum^*_{\alpha<\kappa}$, where κ is an infinite regular cardinal and $\forall \alpha < \kappa : |\{\beta < \kappa | \varphi_\alpha \leqslant \varphi_\beta\}| = \kappa$.

Theorem (Laver [1973])

The class of scattered additively indecomposable linear orderings is the closure of $\{0, 1\}$ under regular unbounded sums.

Theorem (Ervin, Marcone, W., 2024)

If φ is a divisible untranscendable linear ordering which is scattered or countable, then $\varphi = 2$.

Definition

A linear ordering is *σ-scattered* if it can be presented as a countable union of scattered orderings.

Conjecture

If φ is a σ-scattered divisible untranscendable linear ordering, then $\varphi = 2$.

Definition

BE (**Binary Exceptionalism**) states that 2 is the only divisible transcendable linear ordering φ

Conjecture

$\mathrm{ZF} + \mathrm{BE}$ *is consistent.*

Definition

The *ordering principle* O states that every set can be linearly ordered.

Theorem (Sierpiński [1947])

O *implies the existence of a Lebesgue-nonmeasurable set.*

Corollary

AD *implies the failure of* O*.*

Conjecture

$\mathrm{ZF} + \mathrm{O} + \mathrm{BE}$ *is consistent.*

Definition

A *Bernstein set* is an $X \subset \mathbb{R}$ such that neither X nor $\mathbb{R} \setminus X$ contain a perfect nonempty subset.

Observation

$\mathbb{R}$ is divisible if and only if there is a Bernstein set.

Question

Does $\mathsf{ZF} + \mathsf{O}$ *imply that there is a Bernstein set?*

Theorem (Siksek [2016])

Every natural number besides
$15, 22, 23, 50, 114, 167, 175, 186, 212, 238, 239, 303, 364, 420, 428, 454$
is the sum of at most seven positive cubes.

Moral

Never let exceptions keep you from proving a theorem!

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