Cichoń's maximum with evasion number

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Table

Cichoń's diagram

Cichoń's diagram is a table of representative cardinal invariants.

$$\begin{array}{ccccccccc}
 & & \mathrm{cov}(\mathcal{N}) & \longrightarrow & \mathrm{non}(\mathcal{M}) & \longrightarrow & \mathrm{cof}(\mathcal{M}) & \longrightarrow & \mathrm{cof}(\mathcal{N}) & \longrightarrow & 2^{\aleph_0} \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & & & \mathfrak{b} & \longrightarrow & \mathfrak{d} & & & & \\
 & & & & \uparrow & & \uparrow & & & & \\
\aleph_1 & \longrightarrow & \mathrm{add}(\mathcal{N}) & \longrightarrow & \mathrm{add}(\mathcal{M}) & \longrightarrow & \mathrm{cov}(\mathcal{M}) & \longrightarrow & \mathrm{non}(\mathcal{N}) & &
\end{array}$$

Here, an arrow $\mathfrak{x} \to \mathfrak{y}$ denotes that the inequality $\mathfrak{x} \leq \mathfrak{y}$ holds. Moreover, $\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{M})\}$ and $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\}$ also hold .

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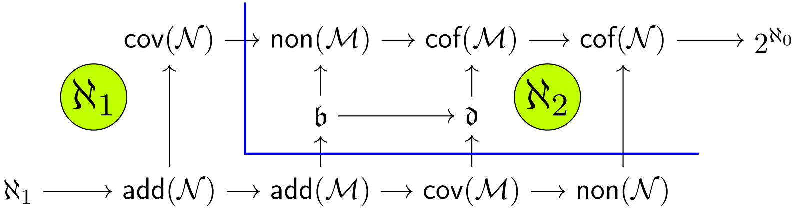
Moreover, it can be *divided into two parts anywhere*.

More precisely, any assignment of $\aleph_1$ and $\aleph_2$ to the numbers in the diagram is consistent whenever it does not conflict with the arrows and the two equations.



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Ultimate Question.

Can we separate Cichoń's diagram with as many values as possible?

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Such a simultaneous separation model is called Cichoń's maximum.

(The ultimate question $\iff$ Does Cichoń's maximum exist?)

Cichoń's maximum

In [GKS19] and [GKMS22] they solved the question positively, constructing Cichoń's maximum with the following order.

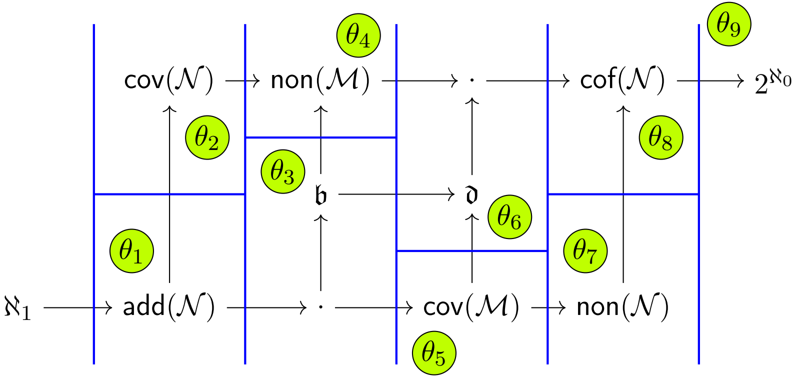

Fig: Cichoń's maximum. add($\mathcal{M}$) and cof($\mathcal{M}$) are omitted as dots "$\cdot$".

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We focus on the *evasion number* $\mathfrak{e}$, one of the classical cardinal invariants.

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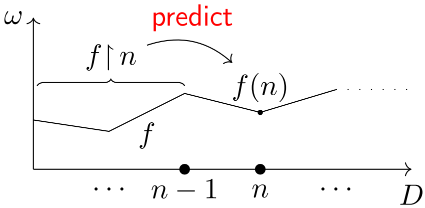
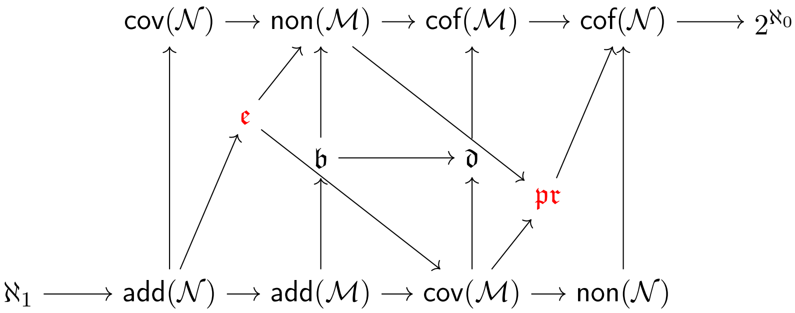
    - $\pi_n : \omega^n \to \omega$, local predictors at each prediction point $n \in D$.
- π predicts $f \in \omega^\omega$
  $:\Leftrightarrow$ for all but finitely many $n \in D$, $f(n) = \pi_n(f \restriction n)$.
- f evades π $:\Leftrightarrow$ π does not predict f.



Evasion/prediction number and Cichoń's diagram

Definition (evasion number $\mathfrak{e}$, prediction number $\mathfrak{pr}$)

- $\mathfrak{pr} := \min \left\{ |\Pi| : \begin{array}{c} \Pi \text{ consists of predictors,} \\ \forall f \in \omega^\omega \ \exists \pi \in \Pi \ \pi \text{ predicts } f \end{array} \right\}.$
- $\mathfrak{e} := \min \{ |F| : F \subseteq \omega^\omega, \forall \text{predictor } \pi \ \exists f \in F \ f \text{ evades } \pi \}.$



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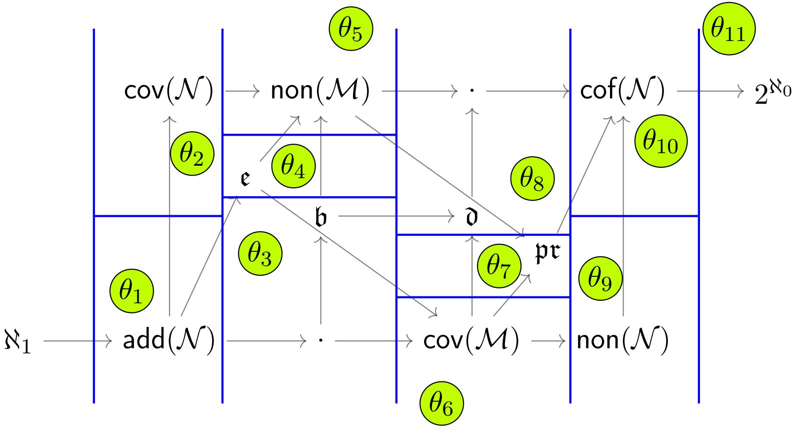
Main Theorem(Y.)

The following separation constellation consistently holds.



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- Our proof is not a modification of the gap. We simply used a different method.

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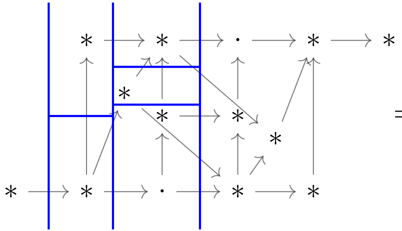
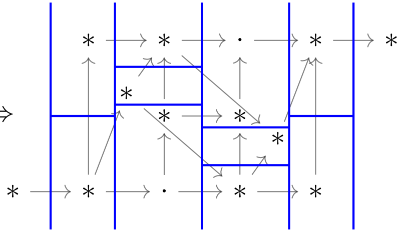
 ⇒ 

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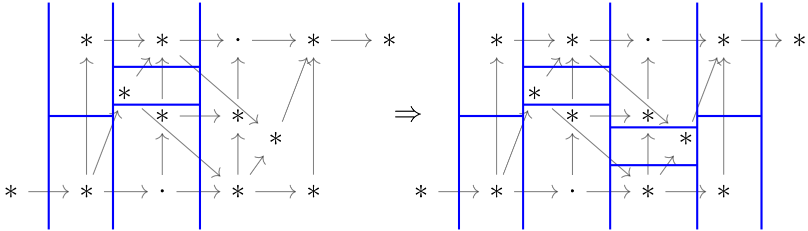

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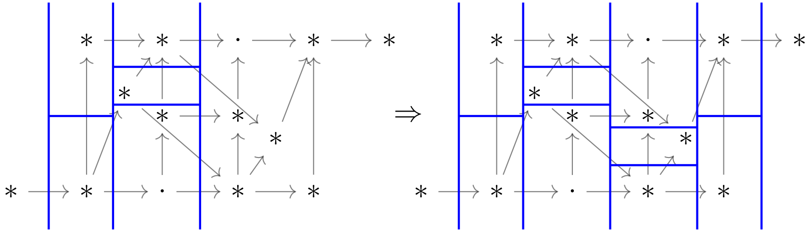

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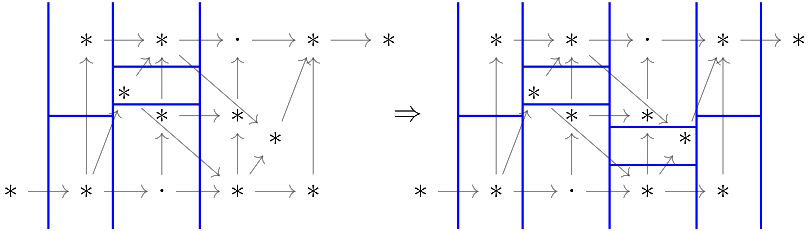
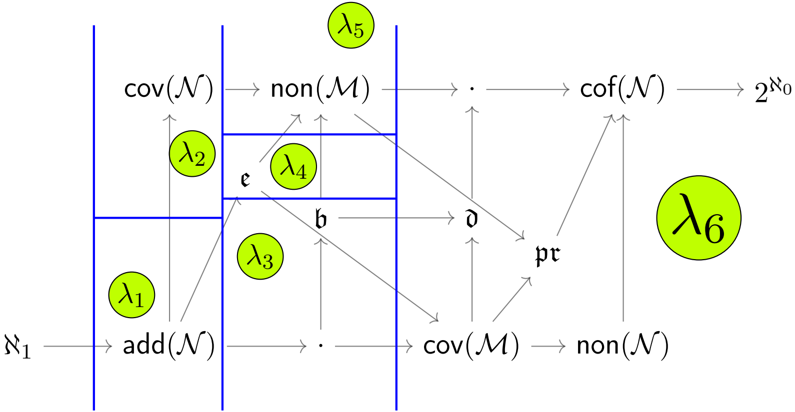


Methods:

- left: finite support iteration of ccc forcings.
- right: Boolean Ultrapowers. [GKS19, with large cardinals]
  submodel method. [GKMS22, without large cardinals]

Separation of the left

Once we separate the left side, the separation of the right is obtained by a simple application of either of the two methods, so the main work is to separate the left side, as follows.



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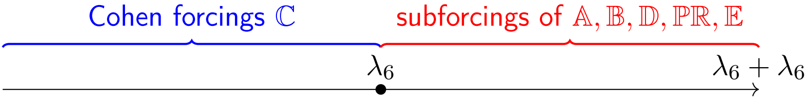

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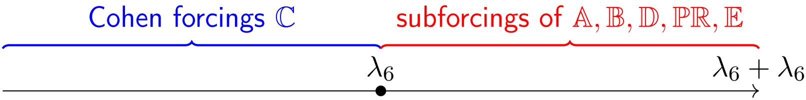
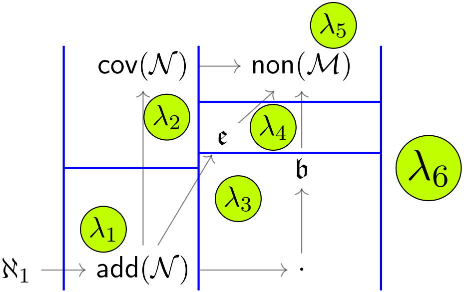


- The first λ_6 iterands are $\mathbb{C}$ (for technical reason).
- Each of the remaining λ_6 iterands is a subforcing of some poset of some size according to the table below left:

poset	size	increase
$\mathbb{A}$	$< \lambda_1$	$\mathrm{add}(\mathcal{N})$
$\mathbb{B}$	$< \lambda_2$	$\mathrm{cov}(\mathcal{N})$
$\mathbb{D}$	$< \lambda_3$	$\mathfrak{b}$
$\mathbb{PR}$	$< \lambda_4$	$\mathfrak{e}$
$\mathbb{E}$	$< \lambda_5$	$\mathrm{non}(\mathcal{M})$



Large and small

Let us see that each number is large ($\mathfrak{x} \geq \lambda_i$) and small ($\mathfrak{x} \leq \lambda_i$).

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For $\mathfrak{e}$, we introduced a new limit notion, *closed-ultrafilter-limit*.

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Example.

Singletons are ultrafilter-limit-linked.
Hence, every $\mathbb{P}$ is $|\mathbb{P}|$-ultrafilter-limit-linked.

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We say $\mathbb{P}_\gamma$ has $< \mu$-ultrafilter-limits if it is an iteration of $< \mu$-ultrafilter-limit-linked forcings and hence we can consider ultrafilter-limits for suitably many sequences (details omitted).

What is "closed"-ultrafilter-limit?

Let $\mathbb{P}$ be a poset (not an iteration).
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Closedness also holds for iteration: If all $\bar{p}^i \in (\mathbb{P}_\gamma)^\omega$ for $i < \omega$ are neatly-lined-up witnessed by some common ∇ and $\dot{Q}_\alpha$ for $\alpha \in \nabla$, then the limits $\langle p^i_\infty = \lim \bar{p}^i \rangle_{i<\omega}$ also form a neatly-lined-up sequence with the same witnesses ∇ and $\dot{Q}_\alpha$ for $\alpha \in \nabla$.

What is "closed"-ultrafilter-limit?

Let $\mathbb{P}$ be a poset (not an iteration).
An uf-lim-linked $Q \subseteq \mathbb{P}$ is closed if $\lim \bar{q} \in Q$ holds for all $\bar{q} \in Q^\omega$.

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We show that this new limit notion helps to control $\mathfrak{e}$.

Main Lemma(Y.)

Closed-ultrafilter-limits keep $\mathfrak{e}$ small.

Linkedness of each iterand of $\mathbb{P}^6$

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(Note: every $\mathbb{P}$ is $|\mathbb{P}|$-closed-uf-lim-linked, splitting into singletons.)

iterand	size	μ-uf-lim-linked	μ-closed-uf-lim-linked
$\mathbb{A}$	$< \lambda_1$	$< \lambda_1$	$< \lambda_1$
$\mathbb{B}$	$< \lambda_2$	$< \lambda_2$	$< \lambda_2$
$\mathbb{D}$	$< \lambda_3$	$< \lambda_3$	$< \lambda_3$
$\mathbb{PR}$	$< \lambda_4$	ω	$< \lambda_4$
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Hence, $\mathbb{P}^6$ has $< \lambda_3$-uf-limits and $< \lambda_4$-closed-uf-limits.

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Hence, $\mathbb{P}^6$ has $< \lambda_3$-uf-limits and $< \lambda_4$-closed-uf-limits.

As a consequence, $\mathbb{P}^6$ forces $\mathfrak{b} \leq \lambda_3$ and $\mathfrak{e} \leq \lambda_4$.
(This is what "$**$-limits keep $\mathfrak{x}$ small" actually means.)

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Sketch of proof. Assume towards contradiction that there is a name $\dot{\pi} = (\dot{D}, \{\dot{\pi}_k : k \in \dot{D}\})$ of a predictor such that some $p \in \mathbb{P}$ forces $\dot{\pi}$ predicts all the λ_4 Cohen reals.

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After some refining argument (e.g. Δ-System Lemma) (and arranging the initial segments of Cohen reals), we obtain $n < \omega$, $s \in \omega^n$, neatly-lined-up sequence $\bar{p} = \langle p_m \rangle_{m<\omega}$ (below p) and countably many Cohen reals $\langle \dot{c}_m \rangle_{m<\omega}$ such that for each $m < \omega$:

$$p_m \Vdash \dot{c}_m \sqsubset_n \dot{\pi} \text{ and } \dot{c}_m{\restriction}(n+1) = s^\frown m,$$

where $f \sqsubset_n \pi :\Leftrightarrow f(k) = \pi_k(f{\restriction}k)$ for all $k \geq n$ in D.

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By the principle of ultrafilter-limit,
$p_\infty \Vdash \exists^\infty m < \omega \ p_m \in \dot{G}$.

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Let $p_\infty := \lim_{m<\omega} p_m$.
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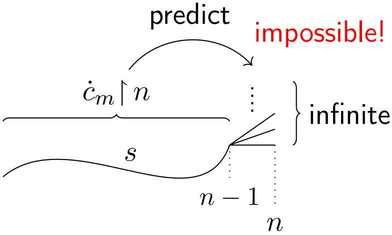

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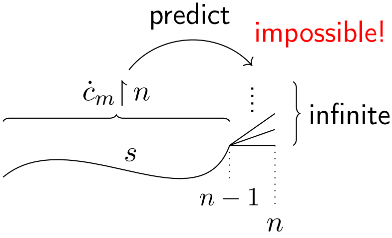

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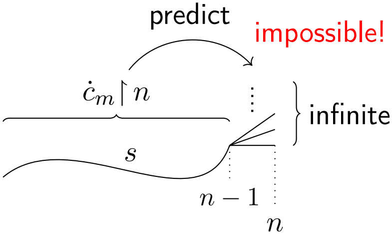

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By arranging the initial segments of the Cohen reals more carefully, we obtain for each $i < \omega$ a limit condition p^i_∞ such that:

$$p^i_\infty \Vdash [n, n+i) \cap \dot{D} = \emptyset,$$

where $[a, b)$ denotes the interval $\{n < \omega : a \leq n < b\}$.

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Thus, we can take the limit of the limit conditions $p^\infty_\infty := \lim_{i<\omega} p^i_\infty$.

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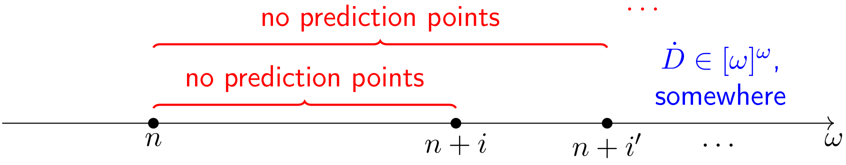
Thus, we can take the limit of the limit conditions $p^\infty_\infty := \lim_{i<\omega} p^i_\infty$. Then, we have:

$$p^\infty_\infty \Vdash [n, n+i) \cap \dot{D} = \emptyset \text{ for infinitely many } i,$$

which contradicts that $\dot{D} \in [\omega]^\omega$ is an infinite set. $\square$



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Other than $\mathfrak{e}$, are there other cardinal invariants which are kept small through forcings with closed-ultrafilter-limits?

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Other than $\mathfrak{e}$, are there other cardinal invariants which are kept small through forcings with closed-ultrafilter-limits?

Question 2.

Is there some mix with the *FAM-limit*, which is another kind of limit notion, focusing on finitely additive measures on ω?

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The preprint was uploaded on the last Monday!