A model with p-measures and no p-point

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1. An ultrafilter $\mathcal{U}$ defines a two valued measure: $\delta_{\mathcal{U}}(A) = 1$ if $A \in \mathcal{U}$ and $\delta_{\mathcal{U}}(A) = 0$ if $A \notin \mathcal{U}$. This is the Dirac delta at $\mathcal{U}$.

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2. Taking suitable $\mathcal{U}$-limits gives another measure: for $A \subseteq \omega$, define $h_A(n) = |A \cap n|/n$. Then

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It is easy to see that $d_{\mathcal{U}}$ extends asymptotic density.

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4. This property is preserved along the iteration by the ω^ω-bounding property of the steps(once a p-measure is killed, it does not resurrect).

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If there is an ultrafilter $\mathcal{U}$ such that $d_{\mathcal{U}}$ is a p-measure, then there is a p-point. In fact, there is $f : \omega \to \omega$ finite to one such that $f(\mathcal{U})$ is a p-point.

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If there is a p-point in the ground model, then after adding any number of random reals there is a p-measure.

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Remark. The measure in the previous theorem still uses a p-point in its definition, although not so direct as in the $\mathcal{U}$-limit.

Theorem(P. Borodulin-Nadzieja, C., A. Morawski)

It is consistent that there is a p-measure, there is no p-point, and $2^{\aleph_0}$ is arbitrarily large. Moreover, it is consitent that no p-measure is a Dirac measure neither an ultrafilter density.

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Forcing with Random over Shelah's model? That's almost the case, we need a natural modification of Shelah's model as in the case of Mekler's model.

Definition(A. Blass, 1986).

Two filters $\mathcal{U}, \mathcal{F}$ are said to be nearly coherent if there is a finite to one function $f : \omega \to \omega$ such that $f(\mathcal{U}) \cup f(\mathcal{F})$ generates a filter.

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2. If $\mathcal{F}_0$ is an ultrafilter an $\mathcal{F}_1$ is just a filter, then they are nearly coherent if and only if there is a finite to one function such that $f(\mathcal{F}_1) \subseteq f(\mathcal{F}_0)$.

Definition(S. Shelah)

Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be two filters on ω. The game $\mathcal{G}(\mathcal{F}_0, \mathcal{F}_1)$ between Player I and Player II is defined as:

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4. At the next stage, Player II answers with $F_j \in [B_j]^{<\omega}$.

Player II wins if and only if $\{a_n : n \in \omega\} \in \mathcal{F}_0$ and $\bigcup_{n\in\omega} F_n \in \mathcal{F}_1$.

Lemma(S. Shelah)

If $\mathcal{F}_0$ is a selective ultrafiler, $\mathcal{F}_1$ is a *p*-point, and they are not nearly coherent, then Player I has no winning strategy in the game $\mathcal{G}(\mathcal{F}_0, \mathcal{F}_1)$.

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This lemma needs still works if instead of $\mathcal{F}_1$ being just *p*-filter instead of a *p*-point.

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In the proof of 3) we use the previous lemma about the game $\mathcal{G}(\mathcal{V}, \mathcal{U})$.

Definition.

Let $\mathbb{B}$ be the random forcing and $\dot{\mathcal{U}}$ a $\mathbb{B}$-name for an ultrafilter. For $p \in \mathbb{B}$, define $\dot{\mathcal{U}}[p] = \{A \subseteq \omega : p \Vdash A \in \dot{\mathcal{U}}\}$.

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Lemma

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Recall that a filter is $\mathcal{F}$ if the quotient $\mathcal{P}(\omega)/\mathcal{F}$ is ccc. Equivalently, if $\mathcal{A} \subseteq \mathcal{F}^+$ is such that for any different $A, B \in \mathcal{A}$ $A \cap B \in \mathcal{F}^*$, then $\mathcal{A}$ is countable.

Proof.

1. Assume otherwise $\dot{\mathcal{U}}[p]$ is not saturated and let $\langle \mathcal{A}_\alpha : \alpha \in \omega_1 \rangle \subseteq \dot{\mathcal{U}}[p]$ be such that for any different $\alpha, \beta \in \omega_1$, $A_\alpha \cap A_\beta \in \dot{\mathcal{U}}[p]^*$.

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4. Second, note that for $\alpha \neq \beta$, q_α and q_β are incompatible, since a common extension would force that $A_\alpha \cap A_\beta \in \dot{\mathcal{U}}$, but $p \Vdash \omega \setminus (A_\alpha \cap A_\beta) \in \dot{\mathcal{U}}$.

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This proves that $\dot{\mathcal{U}}[p]$ is saturated. $\square$

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4. Then we have that $p \Vdash \bigcup_{n\in\omega} A_n \cap g(n) \in \dot{\mathcal{U}}$, so $\bigcup_{n\in\omega} A_n \cap g(n) \in \dot{\mathcal{U}}[p]$

Lemma

Let $\mathcal{U}$ be a selective ultrafilter and $\dot{\mathcal{V}}$ be a $\mathbb{B}$-name such that $\mathbb{B} \Vdash \dot{\mathcal{V}}$ is a p-point. Then for all $p \in \mathbb{B}$, $\mathcal{U}$ and $\dot{\mathcal{V}}[p]$ are not nearly coherent.

Proof.

1 First note that if $\mathcal{F}$ is a filter which is Rudin-Blass above $\mathcal{U}$, witness by f, then $\mathcal{F}$ can not be extended to a p-point after forcing with $\mathbb{B}$: otherwise, if $\dot{\mathcal{V}}$ is a p-point extending $\mathcal{F}$, then $f(\mathcal{V})$ would be a p-point extending $\mathcal{U}$, which is not possible by Kunen's theorem.

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6. By definition of $\dot{\mathcal{V}}[p_0]$, for each $q \leq p$, there is $q \leq p$ such that $q \Vdash h^{-1}[A_p] \notin \dot{\mathcal{V}}$.

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6. By definition of $\dot{\mathcal{V}}[p_0]$, for each $q \leq p$, there is $q \leq p$ such that $q \Vdash h^{-1}[A_p] \notin \dot{\mathcal{V}}$.
7. Then $D = \{p \leq p_0 : (\exists A \in h(\mathcal{U}))(p \Vdash h^{-1}[A] \notin \dot{\mathcal{V}})\}$ is dense below p_0.

□

8 Let $\mathcal{A} \subseteq D$ be a maximal antichain below p_0, and for each $p \in \mathcal{A}$, let $A_p \in h(\mathcal{U})$ be such that $p \Vdash h^{-1}[A_p] \notin \dot{\mathcal{V}}$.

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11. Since $\mathcal{A}$ is a maximal antichain below p_0, it follows that $p_0 \Vdash h^{-1}[X] \notin \dot{\mathcal{V}}$.

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13 Thus, we have $X \in h(\mathcal{U})$ and $\omega \setminus X \in h(\dot{\mathcal{V}}[p_0])$, which contradicts our initial assumption of $h(\dot{\mathcal{V}}[p_0]) \subseteq h(\mathcal{U})$.

Theorem(P. Borodulin-Nadzieja, C., A. Morawski)

If ZFC is consistent, then there is a model such that:

1. There is a *p*-measure.
2. There is no *p*-point.
3. No *p*-measure is a Dirac measure neither an ultrafilter density.
4. $2^{\aleph_0} = \kappa$ for a predetermined regular cardinal $\kappa \geq \omega_2$.

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5. Define a countable support iteration $\mathbb{P}_{\omega_2} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2 \rangle$ as follows:

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   iii) If $\alpha \in S$, and A_α codifies and $\mathbb{P}_\alpha$-name for a saturated p-filter $\dot{\mathcal{F}}$ which is not nearly coherent with $\mathcal{U}_0$, define $\mathbb{P}_\alpha \Vdash \dot{\mathbb{Q}}_\alpha = SP(\dot{\mathcal{F}})$; otherwise, let $\dot{\mathbb{Q}}_\alpha$ be the trivial forcing.

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Then define $\mathbb{P} = \mathbb{P}_{\omega_2} * \dot{\mathbb{B}}_\kappa$. The model is $V[G * H]$, where $G * H$ is $\mathbb{P}$-generic over V.

1. Since $\mathbb{P}_{\omega_2}$ is bounding, proper and preserves $\mathcal{U}_0$, $\mathcal{U}_0$ remains as a selective ultrafilter in $V[G]$, so by P. Borodulin-Nadzieja-Sobota theorem, there is a p-measure in $V[G * H]$.

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7. Then, $\dot{\mathcal{F}}$ can not be a p-point.

Thank you very much!