Function spaces on Corson-like compacta

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Equivalently, a compact space K is an Eberlein compactum if K can be embedded in the following subspace of the product $\mathbb{R}^\Gamma$:

$$c_0(\Gamma) = \{x \in \mathbb{R}^\Gamma : \text{ for every } \varepsilon > 0 \text{ the set } \{\gamma : |x(\gamma)| > \varepsilon\} \text{ is finite}\},$$

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Continuous images, closed subspaces, countable products of Eberlein compacta are Eberlein compact spaces.

A compact space K is Corson compact if, for some set Γ, K is homeomorphic to a subset of the Σ-product of real lines

$$\Sigma(\mathbb{R}^\Gamma) = \{x \in \mathbb{R}^\Gamma : |\{\gamma : x(\gamma) \neq 0\}| \leq \omega\}.$$

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Let κ be an infinite cardinal number. A compact space K is κ-Corson compact if, for some set Γ, K is homeomorphic to a subset of the Σ_κ-product of real lines

$$\Sigma_\kappa(\mathbb{R}^\Gamma) = \{x \in \mathbb{R}^\Gamma : |\{\gamma : x(\gamma) \neq 0\}| < \kappa\}.$$

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Obviously, the class of Corson compact spaces coincides with the class of ω_1-Corson compact spaces.

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If $X_\gamma = I = [0, 1]$ and $a_\gamma = 0$, for all $\gamma \in \Gamma$, then we denote the σ-product $\sigma(X_\gamma, a_\gamma, \Gamma)$ by $\sigma(I, \Gamma)$.

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If $X_\gamma = I^\omega$ and $a_\gamma = (0, 0, \dots)$, for all $\gamma \in \Gamma$, then we denote the σ-product $\sigma(X_\gamma, a_\gamma, \Gamma)$ by $\sigma(I^\omega, \Gamma)$.

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For $\kappa = \omega$, $\Sigma_\kappa(\mathbb{R}^\Gamma) = \sigma(\mathbb{R}, \Gamma)$.

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Proposition

For a compact space K we have

(a) *K is ω-Corson if and only if it can be embedded into some σ-product of metrizable finitely dimensional compacta if and only if it can be embedded into the σ-product $\sigma(I,\Gamma)$ for some set Γ.*

(b) *K is NY compact if and only if it can be embedded into the σ-product $\sigma(I^\omega,\Gamma)$ for some set Γ.*

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Theorem (Marciszewski, Plebanek, Z.)

For a compact space K, the following conditions are equivalent:

- (a) *K is ω-Corson;*
- (b) *K is hereditarily metacompact and each nonempty subspace A of K contains a nonempty relatively open separable, metrizable, finitely dimensional subspace U.*

Theorem (Marciszewski, Plebanek, Z.)

For a compact space K, the following conditions are equivalent:

(a) *K belongs to the class $\mathcal{NY}$;*

(b) *K is hereditarily metacompact and each nonempty subspace A of K contains a nonempty relatively open subspace U of countable weight.*

Theorem (Marciszewski, Plebanek, Z.)

For a compact space K, the following conditions are equivalent:

(a) *K belongs to the class $\mathcal{N}\mathcal{Y}$;*

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Proposition (Nakhmanson and Yakovlev)

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For a space X, $C_p(X)$ denotes the space of real continuous functions on X endowed with the pointwise convergence topology.

Theorem (Z.)

Let K and L be compact spaces. Assume there exists a continuous linear transformation $T : C_p(K) \to C_p(L)$ such that $T(C_p(K))$ is dense in $C_p(L)$. If K is NY compact, then L is NY compact as well.

Definition

For a continuous linear operator $T : X \to Y$ between two linear topological spaces, the dual operator $T^* : X^* \to Y^*$ is given by the formula $T(\phi) = \phi \circ T$.

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Let K and L be compact spaces. Assume there exists a continuous linear transformation $T : C_p(K) \to C_p(L)$ such that $T(C_p(K))$ is dense in $C_p(L)$. If K is NY compact, then L is a union of countably many NY compact spaces.

Lemma

Let K be a compact space such that $K = \bigcup_{n\in\mathbb{N}} K_n$ where $\{K_n : n \in \mathbb{N}\}$ is a sequence of NY compact spaces. Then every subspace of K contains a relatively open subspace of countable weight.

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Proof.

Assume there exists $A \subseteq K$ such that every relatively open $U \subseteq A$ has uncountable weight, then $\overline{A}$ has the same property. Indeed, assume there is a nonempty, relatively open $U \subseteq \overline{A}$ of countable weight, then $U \cap A \neq \emptyset$, and therefore $U \cap A \subseteq A$ is a relatively open, nonempty subset of A with $w(U \cap A) \leq \omega$, contradiction.
Without loss of generality, we can assume that A is closed and therefore compact. Then it has the same property as K, so we can assume that $A = K$. As $K = \bigcup_{n\in\mathbb{N}} K_n$, by the Baire category theorem, there is K_n with nonempty interior. Since K_n is *NY* compact, there exists an open $V \subseteq int_K(K_n)$ with $w(V) \leq \omega$. Then V is an open subset of K of countable weight. □

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Theorem (Gruenhage)

For a compact space K, the following conditions are equivalent:
a) K is Eberlein compact;
b) K^2 is hereditarily σ - metacompact.

Lemma

Let K and L be compact spaces. Assume there exists a continuous linear transformation $T : C_p(K) \to C_p(L)$ such that $T(C_p(K))$ is dense in $C_p(L)$. If K is NY compact, then L is hereditarily σ - metacompact.

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Proof.

Space K is Eberlein compact, so there exists a dense σ - compact set $D \subset C_p(K)$. Then $T(D)$ is again a dense σ - compact subset of $C_p(L)$. Consequently, space L is Eberlein compact and therefore hereditarily σ - metacompact. □

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Concluding, space L is hereditarily σ - metacompact and is a union of countably many *NY* compact spaces.

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Concluding, space L is hereditarily σ - metacompact and is a union of countably many *NY* compact spaces.

Lemma

A σ - metacompact space which is a union of countably many closed, metacompact subspaces is metacompact.

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Theorem (Marciszewski, Plebanek, Z.)

An NY compact space K is ω-Corson compact if and only if it is strongly countably dimensional.

Theorem (Z.)

Let X and Y be σ-compact spaces. Assume there exists a continuous linear transformation $T : C_p(X) \to C_p(Y)$ such that $T(C_p(X))$ is dense in $C_p(Y)$. If X is strongly countably dimensional, then Y is strongly countably dimensional as well.

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Theorem (Z.)

Let K and L be compact spaces. Assume there exists a continuous linear transformation $T : C_p(K) \to C_p(L)$ such that $T(C_p(K))$ is dense in $C_p(L)$. If K is ω - Corson compact, then L is ω - Corson compact as well.