Auerbach systems and Lusin sets

Kamil Ryduchowski
under supervision of Piotr Koszmider

Institute of Mathematics of the Polish Academy of Sciences
Faculty of Mathematics, Informatics and Mechanics, University of Warsaw

WINTER SCHOOL IN ABSTRACT ANALYSIS 2024

Definition

An Auerbach system of size κ in a Banach space X is a sequence $(u_\alpha, g_\alpha)_{\alpha<\kappa}$ such that:

- $u_\alpha \in X$, $\|u_\alpha\| = 1$ for every $\alpha < \kappa$
- $g_\alpha \in X^*$, $\|g_\alpha\| = 1$ for every $\alpha < \kappa$

Definition

An Auerbach system of size κ in a Banach space X is a sequence $(u_\alpha, g_\alpha)_{\alpha<\kappa}$ such that:

- $u_\alpha \in X$, $\|u_\alpha\| = 1$ for every $\alpha < \kappa$
- $g_\alpha \in X^*$, $\|g_\alpha\| = 1$ for every $\alpha < \kappa$
- $g_\alpha(u_\beta) = \delta_{\alpha,\beta}$ for all $\alpha, \beta < \kappa$

Theorem (Hájek, Kania, Russo)

Assume CH. Then there is an equivalent renorming of $c_0(\omega_1)$ without uncountable Auerbach systems.

Theorem (Hájek, Kania, Russo)

Assume CH. Then there is an equivalent renorming of $c_0(\omega_1)$ without uncountable Auerbach systems.

Is it true in ZFC or is the negation consistent?

Theorem (Hájek, Kania, Russo)

Assume CH. Then there is an equivalent renorming of $c_0(\omega_1)$ without uncountable Auerbach systems.

Is it true in ZFC or is the negation consistent?
We will show that the statement is true provided that either of the following holds:

- $cf(2^\omega) = \omega_1$

Theorem (Hájek, Kania, Russo)

Assume CH. Then there is an equivalent renorming of $c_0(\omega_1)$ without uncountable Auerbach systems.

Is it true in ZFC or is the negation consistent?
We will show that the statement is true provided that either of the following holds:

- $cf(2^\omega) = \omega_1$
- There is a strongly Lusin set, i.e. an uncountable set $L \subseteq \mathbb{R}$ such that for any sequence $(\lambda_1^\alpha, \ldots, \lambda_n^\alpha)_{\alpha<\omega_1}$ of pairwise disjoint n-tuples of elements of L (without repetitions) and any meager set $M \subseteq \mathbb{R}^n$ the intersection

$$M \cap \{(\lambda_1^\alpha, \ldots, \lambda_n^\alpha) : \alpha < \omega_1\}$$

is countable.

For $x \in c_0(\omega_1)$ we will put

$$\|x\| = \sup_{\alpha<\omega_1} |\varphi_\alpha(x)|.$$

For $x \in c_0(\omega_1)$ we will put

$$\|x\| = \sup_{\alpha<\omega_1} |\varphi_\alpha(x)|.$$

For $\alpha < \omega_1$ fix an injection $e_\alpha \colon \alpha + 1 \to \omega$ so that $e_\alpha(\alpha) = 0$ and $e_\alpha =^* e_\beta|_{\alpha+1}$ for $\alpha < \beta < \omega_1$.

For $x \in c_0(\omega_1)$ we will put

$$\|x\| = \sup_{\alpha<\omega_1} |\varphi_\alpha(x)|.$$

For $\alpha < \omega_1$ fix an injection $e_\alpha \colon \alpha + 1 \to \omega$ so that $e_\alpha(\alpha) = 0$ and $e_\alpha =^* e_\beta|_{\alpha+1}$ for $\alpha < \beta < \omega_1$.
Fix ω_1 pairwise distinct real numbers $(\lambda_\alpha)_{\alpha<\omega_1}$ from some small interval $(0, \varepsilon)$.

For $x \in c_0(\omega_1)$ we will put

$$\|x\| = \sup_{\alpha<\omega_1} |\varphi_\alpha(x)|.$$

For $\alpha < \omega_1$ fix an injection $e_\alpha \colon \alpha + 1 \to \omega$ so that $e_\alpha(\alpha) = 0$ and $e_\alpha =^* e_\beta|_{\alpha+1}$ for $\alpha < \beta < \omega_1$.
Fix ω_1 pairwise distinct real numbers $(\lambda_\alpha)_{\alpha<\omega_1}$ from some small interval $(0, \varepsilon)$.
We define φ_α by its ℓ_1-representation

$$\varphi_\alpha(\xi) = \lambda_\alpha^{e_\alpha(\xi)} \text{ for } \xi \leq \alpha,$$

$$\varphi_\alpha(\xi) = 0 \text{ for } \alpha < \xi < \omega_1.$$

Assume that there is an Auerbach system $(u_\alpha, g_\alpha)_{\alpha<\omega_1}$ in the space $(c_0(\omega_1), \|\cdot\|)$. Then g_α is a **finite** linear combinations of $(\varphi_\alpha)_{\alpha<\omega_1}$, say

$$g_\alpha = \sum_{i=1}^{N} c_i^\alpha \varphi_{\beta_i^\alpha}.$$

Assume that there is an Auerbach system $(u_\alpha, g_\alpha)_{\alpha<\omega_1}$ in the space $(c_0(\omega_1), \|\cdot\|)$. Then g_α is a **finite** linear combinations of $(\varphi_\alpha)_{\alpha<\omega_1}$, say

$$g_\alpha = \sum_{i=1}^{N} c_i^\alpha \varphi_{\beta_i^\alpha}.$$

Furthermore, we may assume that either:
(1): $\{\beta_1^\alpha, \ldots, \beta_N^\alpha\} < \{\beta_1^\gamma, \ldots, \beta_N^\gamma\}$ for $\alpha < \gamma < \omega_1$

Assume that there is an Auerbach system $(u_\alpha, g_\alpha)_{\alpha<\omega_1}$ in the space $(c_0(\omega_1), \|\cdot\|)$. Then g_α is a **finite** linear combinations of $(\varphi_\alpha)_{\alpha<\omega_1}$, say

$$g_\alpha = \sum_{i=1}^{N} c_i^\alpha \varphi_{\beta_i^\alpha}.$$

Furthermore, we may assume that either:
(1): $\{\beta_1^\alpha, \ldots, \beta_N^\alpha\} < \{\beta_1^\gamma, \ldots, \beta_N^\gamma\}$ for $\alpha < \gamma < \omega_1$
or
(2): $\beta_1^\alpha = \delta$ for every $\alpha < \omega_1$ and $\{\beta_2^\alpha, \ldots, \beta_N^\alpha\} < \{\beta_2^\gamma, \ldots, \beta_N^\gamma\}$ for $\alpha < \gamma < \omega_1$.

Assume that there is an Auerbach system $(u_\alpha, g_\alpha)_{\alpha<\omega_1}$ in the space $(c_0(\omega_1), \|\cdot\|)$. Then g_α is a **finite** linear combinations of $(\varphi_\alpha)_{\alpha<\omega_1}$, say

$$g_\alpha = \sum_{i=1}^{N} c_i^\alpha \varphi_{\beta_i^\alpha}.$$

Furthermore, we may assume that either:
(1): $\{\beta_1^\alpha, \ldots, \beta_N^\alpha\} < \{\beta_1^\gamma, \ldots, \beta_N^\gamma\}$ for $\alpha < \gamma < \omega_1$
or
(2): $\beta_1^\alpha = \delta$ for every $\alpha < \omega_1$ and $\{\beta_2^\alpha, \ldots, \beta_N^\alpha\} < \{\beta_2^\gamma, \ldots, \beta_N^\gamma\}$ for $\alpha < \gamma < \omega_1$.
For our discussion today, focus on (1).

The very general idea is as follows: For every $\alpha \in (0, \omega_1)$ we have

$$g_\alpha(u_0) = 0.$$

The very general idea is as follows: For every $\alpha \in (0, \omega_1)$ we have

$$g_\alpha(u_0) = 0.$$

But that means that for uncountably many αs

$$\sum_{i=1}^{N} c_i^\alpha \Big(\sum_{\xi<\alpha} \lambda_{\beta_i^\alpha}^{e_{\beta_i^\alpha}(\xi)} u_0(\xi) \Big) = 0.$$

Basic idea

The very general idea is as follows: For every $\alpha \in (0, \omega_1)$ we have

$$g_\alpha(u_0) = 0.$$

But that means that for uncountably many αs

$$\sum_{i=1}^{N} c_i^\alpha \Big(\sum_{\xi<\alpha} \lambda_{\beta_i^\alpha}^{e_{\beta_i^\alpha}(\xi)} u_0(\xi) \Big) = 0.$$

But **surely** we may pick uncountably many numbers λ_α in a way so that some of them don't satisfy that equation. *Right?*

In fact we work with N equations,

$$g_\alpha(u_1) = g_\alpha(u_2) = \ldots = g_\alpha(u_N) = 0.$$

The devil is in the details

In fact we work with N equations,

$$g_\alpha(u_1) = g_\alpha(u_2) = \ldots = g_\alpha(u_N) = 0.$$

Let $A = supp(u_1) \cup \ldots \cup supp(u_N)$ and find $f_i \colon A \to \mathbb{N}$ such that $e_{\beta_i^\alpha}|_A = f_i$ for uncountably many αs.

In fact we work with N equations,

$$g_\alpha(u_1) = g_\alpha(u_2) = \ldots = g_\alpha(u_N) = 0.$$

Let $A = supp(u_1) \cup \ldots \cup supp(u_N)$ and find $f_i \colon A \to \mathbb{N}$ such that $e_{\beta_i^\alpha}|_A = f_i$ for uncountably many αs.
Write those equations as

$$\begin{bmatrix} \sum_{\xi\in A} \lambda_{\beta_1^\alpha}^{f_1(\xi)} u_1(\xi) & \sum_{\xi\in A} \lambda_{\beta_2^\alpha}^{f_2(\xi)} u_1(\xi) & \ldots & \sum_{\xi\in A} \lambda_{\beta_N^\alpha}^{f_N(\xi)} u_1(\xi) \\ \sum_{\xi\in A} \lambda_{\beta_1^\alpha}^{f_1(\xi)} u_2(\xi) & \sum_{\xi\in A} \lambda_{\beta_2^\alpha}^{f_2(\xi)} u_2(\xi) & \ldots & \sum_{\xi\in A} \lambda_{\beta_N^\alpha}^{f_N(\xi)} u_2(\xi) \\ \ldots & & & \\ \sum_{\xi\in A} \lambda_{\beta_1^\alpha}^{f_1(\xi)} u_N(\xi) & \sum_{\xi\in A} \lambda_{\beta_2^\alpha}^{f_2(\xi)} u_N(\xi) & \ldots & \sum_{\xi\in A} \lambda_{\beta_N^\alpha}^{f_N(\xi)} u_N(\xi) \end{bmatrix} \begin{bmatrix} c_1^\alpha \\ c_2^\alpha \\ \ldots \\ c_N^\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ \ldots \\ 0 \end{bmatrix}.$$

Someone stop these indices!

Consider $h\colon (0,\varepsilon)^N \to \mathbb{R}$ given by

$$h(x_1, x_2, \ldots, x_N) = det \begin{bmatrix} \sum_{\xi \in A} x_1^{f_1(\xi)} u_1(\xi) & \ldots & \sum_{\xi \in A} x_N^{f_N(\xi)} u_1(\xi) \\ \sum_{\xi \in A} x_1^{f_1(\xi)} u_2(\xi) & \ldots & \sum_{\xi \in A} x_N^{f_N(\xi)} u_2(\xi) \\ & \ldots & \\ \sum_{\xi \in A} x_1^{f_1(\xi)} u_N(\xi) & \ldots & \sum_{\xi \in A} x_N^{f_N(\xi)} u_N(\xi) \end{bmatrix}$$

Consider $h\colon (0,\varepsilon)^N \to \mathbb{R}$ given by

$$h(x_1, x_2, \ldots, x_N) = det \begin{bmatrix} \sum_{\xi \in A} x_1^{f_1(\xi)} u_1(\xi) & \ldots & \sum_{\xi \in A} x_N^{f_N(\xi)} u_1(\xi) \\ \sum_{\xi \in A} x_1^{f_1(\xi)} u_2(\xi) & \ldots & \sum_{\xi \in A} x_N^{f_N(\xi)} u_2(\xi) \\ & \ldots & \\ \sum_{\xi \in A} x_1^{f_1(\xi)} u_N(\xi) & \ldots & \sum_{\xi \in A} x_N^{f_N(\xi)} u_N(\xi) \end{bmatrix}$$

Let us denote by ψ_i^x, where $x \in (0,\delta)$ and $i = 1, \ldots, N$, the functional on $c_0(\omega_1)$ given by its $\ell_1(\omega_1)$ representation: $\psi_i^x(\xi) = x^{f_i(\xi)}$ for $\xi \in A$, $\psi_i^x(\xi) = 0$ otherwise.

Consider $h\colon (0,\varepsilon)^N \to \mathbb{R}$ given by

$$h(x_1, x_2, \ldots, x_N) = det \begin{bmatrix} \sum_{\xi\in A} x_1^{f_1(\xi)} u_1(\xi) & \ldots & \sum_{\xi\in A} x_N^{f_N(\xi)} u_1(\xi) \\ \sum_{\xi\in A} x_1^{f_1(\xi)} u_2(\xi) & \ldots & \sum_{\xi\in A} x_N^{f_N(\xi)} u_2(\xi) \\ \ldots \\ \sum_{\xi\in A} x_1^{f_1(\xi)} u_N(\xi) & \ldots & \sum_{\xi\in A} x_N^{f_N(\xi)} u_N(\xi) \end{bmatrix}$$

Let us denote by ψ_i^x, where $x \in (0, \delta)$ and $i = 1, \ldots, N$, the functional on $c_0(\omega_1)$ given by its $\ell_1(\omega_1)$ representation: $\psi_i^x(\xi) = x^{f_i(\xi)}$ for $\xi \in A$, $\psi_i^x(\xi) = 0$ otherwise.
Clearly

$$h(x_1, x_2, \ldots, x_N) = det \begin{bmatrix} \psi_1^{x_1}(u_1) & \psi_2^{x_2}(u_1) & \ldots & \psi_N^{x_N}(u_1) \\ \psi_1^{x_1}(u_2) & \psi_2^{x_2}(u_2) & \ldots & \psi_N^{x_N}(u_2) \\ \ldots \\ \psi_1^{x_1}(u_N) & \psi_2^{x_2}(u_N) & \ldots & \psi_N^{x_N}(u_N) \end{bmatrix}.$$

Let us consider such determinants for small N. Start with $N = 1$:

$$\psi_1^{x_1}(u_1) = \sum_{\xi \in A} x_1^{f_1(\xi)} u_1(\xi)$$

Let us consider such determinants for small N. Start with $N = 1$:

$$\psi_1^{x_1}(u_1) = \sum_{\xi \in A} x_1^{f_1(\xi)} u_1(\xi)$$

By the definition this is a power series with non-zero coefficients, so there is a cofinite (in particular: comeager) set $J_1 \subseteq (0, \varepsilon)$ such that it is non-zero for $x_1 \in J_1$.

$$\psi_1^{x_1}(u_1)\cdot\psi_2^{x_2}(u_2)-\psi_1^{x_1}(u_2)\cdot\psi_2^{x_2}(u_1)=\psi_2^{x_2}\Big(\psi_1^{x_1}(u_1)u_2-\psi_1^{x_1}(u_2)u_1\Big).$$

$$\psi_1^{x_1}(u_1)\cdot\psi_2^{x_2}(u_2)-\psi_1^{x_1}(u_2)\cdot\psi_2^{x_2}(u_1)=\psi_2^{x_2}\Big(\psi_1^{x_1}(u_1)u_2-\psi_1^{x_1}(u_2)u_1\Big).$$

If $x_1 \in J_1$, then the latter is the evaluation of a non-zero functional $\psi_2^{x_2}$ on a non-zero vector (as a nontrivial combination of linearly independent vectors), so we may repeat the argument with the power series.

$$\psi_1^{x_1}(u_1)\cdot\psi_2^{x_2}(u_2)-\psi_1^{x_1}(u_2)\cdot\psi_2^{x_2}(u_1)=\psi_2^{x_2}\left(\psi_1^{x_1}(u_1)u_2-\psi_1^{x_1}(u_2)u_1\right).$$

If $x_1\in J_1$, then the latter is the evaluation of a non-zero functional $\psi_2^{x_2}$ on a non-zero vector (as a nontrivial combination of linearly independent vectors), so we may repeat the argument with the power series. Therefore for every $x_1\in J_1$ there is a cofinite set $K(x_1)\subseteq(0,\varepsilon)$ such that h_2 is non-zero on (x_1,x_2) for $x_2\in K(x_1)$.

$$\psi_1^{x_1}(u_1) \cdot \psi_2^{x_2}(u_2) - \psi_1^{x_1}(u_2) \cdot \psi_2^{x_2}(u_1) = \psi_2^{x_2}\Big(\psi_1^{x_1}(u_1)u_2 - \psi_1^{x_1}(u_2)u_1\Big).$$

If $x_1 \in J_1$, then the latter is the evaluation of a non-zero functional $\psi_2^{x_2}$ on a non-zero vector (as a nontrivial combination of linearly independent vectors), so we may repeat the argument with the power series. Therefore for every $x_1 \in J_1$ there is a cofinite set $K(x_1) \subseteq (0, \varepsilon)$ such that h_2 is non-zero on (x_1, x_2) for $x_2 \in K(x_1)$. By the Kuratowski-Ulam theorem, there is a comeager set $M \subseteq (0, \delta)^2$ such that $h_2(x_1, x_2) \neq 0$ for $(x_1, x_2) \in M$.

$$\psi_1^{x_1}(u_1)\cdot\psi_2^{x_2}(u_2)-\psi_1^{x_1}(u_2)\cdot\psi_2^{x_2}(u_1)=\psi_2^{x_2}\Big(\psi_1^{x_1}(u_1)u_2-\psi_1^{x_1}(u_2)u_1\Big).$$

If $x_1 \in J_1$, then the latter is the evaluation of a non-zero functional $\psi_2^{x_2}$ on a non-zero vector (as a nontrivial combination of linearly independent vectors), so we may repeat the argument with the power series. Therefore for every $x_1 \in J_1$ there is a cofinite set $K(x_1) \subseteq (0, \varepsilon)$ such that h_2 is non-zero on (x_1, x_2) for $x_2 \in K(x_1)$. By the Kuratowski-Ulam theorem, there is a comeager set $M \subseteq (0, \delta)^2$ such that $h_2(x_1, x_2) \neq 0$ for $(x_1, x_2) \in M$.
Proceeding by induction we show that there is a comeager set $K \subseteq (0, \delta)^N$ such that $h(x_1, \ldots, x_n) \neq 0$ for $(x_1, \ldots, x_n) \in K$.
Therefore if $\{\lambda_\alpha : \alpha < \omega_1\}$ is strongly Lusin, we may pick $(\lambda_{\beta_1^\alpha}, \lambda_{\beta_2^\alpha}, \ldots, \lambda_{\beta_N^\alpha}) \in K$.

- Do we need extra set-theoretic assumptions?[1]

[1] To solve this problem. We obviously need them to keep our jobs.

- Do we need extra set-theoretic assumptions?[1]
- Study the structure of the set $\{(x_1, \ldots, x_n) \in (0, \varepsilon)^N : \text{ that determinant is zero}\}$.

[1] To solve this problem. We obviously need them to keep our jobs.

- Do we need extra set-theoretic assumptions?[1]
- Study the structure of the set $\{(x_1, \dots, x_n) \in (0, \varepsilon)^N : \text{ that determinant is zero}\}$.
- Is every $A \in \big(Fin(\mathbb{R}) \otimes Fin(\mathbb{R})\big) \otimes \dots \otimes Fin(\mathbb{R})$ covered by a set of such form?

[1] To solve this problem. We obviously need them to keep our jobs.

- Do we need extra set-theoretic assumptions?[1]
- Study the structure of the set $\{(x_1, \dots, x_n) \in (0, \varepsilon)^N : \text{ that determinant is zero}\}$.
- Is every $A \in \big(Fin(\mathbb{R}) \otimes Fin(\mathbb{R})\big) \otimes \dots \otimes Fin(\mathbb{R})$ covered by a set of such form?
- Is there (in ZFC) a version of a strongly Lusin set for Fubini products of $Fin(\mathbb{R})$?

[1] To solve this problem. We obviously need them to keep our jobs.

- Do we need extra set-theoretic assumptions?[1]
- Study the structure of the set $\{(x_1, \ldots, x_n) \in (0, \varepsilon)^N : \text{ that determinant is zero}\}$.
- Is every $A \in \big(Fin(\mathbb{R}) \otimes Fin(\mathbb{R})\big) \otimes \ldots \otimes Fin(\mathbb{R})$ covered by a set of such form?
- Is there (in ZFC) a version of a strongly Lusin set for Fubini products of $Fin(\mathbb{R})$?
- How even start a construction of a ccc forcing notion which adds an Auerbach system?

[1] To solve this problem. We obviously need them to keep our jobs.