Non-meager filters

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Overview

1. Types of filters
2. Hierarchy of filters
3. Filters and forcing

Types of filters

For a filter $\mathcal{F}$ we define $\mathcal{P}(\omega)/\mathcal{F}$ as the quotient algebra of the equivalence relation defined by

$$A \sim B \iff A \triangle B \in \mathcal{F}^c,$$

where $\mathcal{F}^c$ denotes the ideal dual to the filter $\mathcal{F}$, i.e.

$$\mathcal{F}^c = \{I \subseteq \omega \colon I^c \in \mathcal{F}\}$$

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Types of filters

- A filter $\mathcal{F}$ **supports a measure** if $\mathcal{F} = \{A : \mu(A) = 1\}$ for some probability measure μ on ω.
- A filter $\mathcal{F}$ is **ccc** if $\mathcal{P}(\omega)/\mathcal{F}$ is ccc.
- A filter $\mathcal{F}$ is **non-meager** if it is non-meager as a subset of 2^ω
- A filter $\mathcal{F}$ is **Fréchet** if it contains all cofinite subsets of ω.

Ultrafilers and filters supporting measure

Theorem

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Proof

Take any ultrafilter $\mathcal{U}$ and measure μ on ω such that:

- $\mu(A) = 0$ iff $A \notin \mathcal{U}$
- $\mu(A) = 1$ iff $A \in \mathcal{U}$

With this measure we can write $\mathcal{U}$ as $\{A : \mu(A) = 1\}$, so $\mathcal{U}$ is a filter supporting measure.

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Then, the family $\mathcal{F}$ is a filter, but clearly not an ultrafilter.

Filters supporting measure and ccc filters

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Proof

Since $\mathcal{F}$ supports measure, then there exists μ such that

$$\mathcal{F} = \{A : \mu(A) = 1\}.$$

So we have that the elements of the dual ideal are the sets of measure 0. Suppose, towards the contradiction, that we can find an uncountable family $(A_\alpha)_{\alpha<\omega_1}$ of subsets of ω such that $\mu(A_\alpha \triangle A_\beta) = 0$ for each $\alpha \neq \beta$ and $\mu(A_\alpha) > 0$ for each $\alpha < \omega_1$. Without loss of generality, passing to an uncountable subfamily if needed, we may assume that there is $a > 0$ such that $\mu(A_\alpha) > a$ for each α. This is a contradiction as $\mu(A_0 \cup \cdots \cup A_m) > 1$ for $m > 1/a$.

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First, note that there is a complete Boolean algebra $\mathbb{A}$ of size $\mathfrak{c}$ which is ccc but which does not support a measure (i.e. there is no measure μ such that $\mu(A) > 0$ for each nonzero element A of $\mathbb{A}$), e.g. the Gaifman algebra.

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We claim that there is a homomorphism $\varphi\colon \mathcal{P}(\omega) \to \mathbb{A}$ which is onto.
This follows from the Fichtencholz-Kantorowicz theorem saying that there is an independent family $(X_\alpha)_{\alpha<\mathfrak{c}}$ of subsets of ω. We can define a function $f\colon \{X_\alpha\colon \alpha < \mathfrak{c}\} \to \mathbb{A}$ which is onto. Then, by Sikorski extension theorem, we can extend f to a homomorphism $\varphi\colon \mathcal{P}(\omega) \to \mathbb{A}$.
Let $\mathcal{F}$ be the filter dual to the kernel of φ. Then $\mathcal{P}(\omega)/\mathcal{F}$ is isomorphic to $\mathbb{A}$ and so $\mathcal{F}$ is a ccc filter which does not support a measure.

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A filter $\mathcal{F}$ is meager if and only if there is an interval partition (I_n) such that for each infinite N we have $\bigcup_{n \in N} I_n \notin \mathcal{F}^c$.

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Proposition

There is a non-meager filter which is not ccc.

Hierarchy of filters

Corollary

$$\text{ultra} \implies \text{supporting measure} \implies \text{ccc} \implies \text{non-meager.}$$

Filters and forcing

We will force with a complete Boolean algebra $\mathbb{A}$. Let G be an $\mathbb{A}$-generic. By $\dot{\mathcal{U}}$ we will denote an $\mathbb{A}$-name for an non-principal ultrafilter in $\mathcal{P}(\omega) \cap V[G]$.

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Suppose $\dot{\mathcal{U}}$ is such that $1 \Vdash \dot{\mathcal{U}}$ is non-principal. Then $1 \Vdash \dot{\mathcal{U}}$ extends the filter consisting of co-finite subsets of ω.

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Example

Consider the forcing with $\mathcal{P}(\omega)/\mathit{fin}$. This forcing adds generically an ultrafilter. Consider the name

$$\dot{\mathcal{U}} = \{\langle A, A\rangle \colon A \in \mathcal{P}(\omega)/\mathit{fin}\}.$$

A generic 'reads' this name as an ultrafilter on $\mathcal{P}(\omega)$. Notice that if $A \in \mathcal{P}(\omega) \cap V$ is a co-infinite set, then $A^c \Vdash A \notin \dot{\mathcal{U}}$. So, there is no ground model filter $\mathcal{F}$ bigger than the Frechet filter for which $1 \Vdash \dot{\mathcal{U}}$ extends $\mathcal{F}$.

Filters and forcing

Theorem

Let $\dot{\mathcal{U}}$ be as above. There exists a filter $\mathcal{F}$ on ω in V such that:

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- there exists an injective Boolean homomorphism $\psi\colon \mathcal{P}(\omega)/\mathcal{F} \to \mathbb{A}$.

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If $\mathbb{A}$ is *ccc*, then every ultrafilter from $\mathcal{P}(\omega) \cap V[G]$ extends a ccc filter $\mathcal{F}$ from the ground model.

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Corollary

If $\mathbb{B}$ is the random forcing and G is a $\mathbb{B}$-generic, then every ultrafilter from $V[G]$ extends a measure supporting filter $\mathcal{F}$ from ground model.

Filters and forcing

Proposition

For an $\mathbb{A}$-name $\dot{\mathcal{U}}$ for an ultrafilter let $\phi\colon \mathcal{P}(\omega) \to \mathbb{A}$ be defined by $\phi(A) = [\![A \in \dot{\mathcal{U}}]\!]$. Then ϕ is a Boolean homomorphism.

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Proof of the theorem

Let ϕ be the homomorphism promised by the above proposition.
Let

$$\mathcal{F} = \{F \in \mathcal{P}(\omega) \cap V : \phi(F) = 1\}.$$

Notice that $\mathcal{F}$ is a filter on ω (as ϕ is a Boolean homomorphism).

Claim. $1 \Vdash \dot{\mathcal{U}}$ extends $\mathcal{F}$.

Indeed, if $\phi(F) = 1$, then $[\![F \in \dot{\mathcal{U}}]\!] = 1$ and so $1 \Vdash F \in \dot{\mathcal{U}}$.

Filters and forcing

Claim. There exists an injective Boolean homomorphism

$$\psi\colon \mathcal{P}(\omega)/\mathcal{F} \to \mathbb{A}.$$

Define $\psi\colon \mathcal{P}(\omega)/\mathcal{F} \to \mathbb{A}$ by

$$\psi([A]_{\mathcal{F}}) = \phi(A).$$

Since ϕ is homomorphism we only have to check that ψ is well defined and it is injective.

- ψ is well defined
  Take A, B such that $[A]_{\mathcal{F}} = [B]_{\mathcal{F}}$. Then $A \Delta B \in ker(\phi)$, so $\phi(A \Delta B) = 0$. Hence $\phi(A) \Delta \phi(B) = 0$, so $\psi([A]_{\mathcal{F}}) = \psi([B]_{\mathcal{F}})$.
- ψ is injective
  Take $[A]_{\mathcal{F}}, [B]_{\mathcal{F}}$, such that $[A]_{\mathcal{F}} \neq [B]_{\mathcal{F}}$, then $A \Delta B \notin ker(\phi)$ so $\phi(A) \Delta \phi(B) \neq 0$. Hence $\psi([A]_{\mathcal{F}}) \neq \psi([B]_{\mathcal{F}})$.