Structural Ramsey theory and topological dynamics II

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Reminder from first lecture

- Extremely amenable group G:
  Every continuous action of G on a compact space has a fixed point.
- Ultrahomogeneous structure $\mathbb{F}$:
  Every isomorphism between finite substructures of $\mathbb{F}$ extends to an automorphism of $\mathbb{F}$.
- Fraïssé class $\mathcal{K}$:
  Countable class of finite structures with hereditarity, amalgamation and joint embedding property.
- Some Polish, non locally compact groups G are extremely amenable.
- When G closed subgroup of S_∞, then
  $G = \mathrm{Aut}(\mathbb{F})$, $\mathbb{F}$ countable ultrahomogeneous structure.
- The class $\mathcal{K}$ of finite substructures of $\mathbb{F}$ is a Fraïssé class.

Part III

Extreme amenability and the Ramsey property

Ordered ultrahomogeneous structures

Proposition

Let $\mathbb{F}$ be a countable ultrahomogeneous structure, with $\mathrm{Aut}(\mathbb{F})$ extremely amenable. Then there is a linear ordering $<$ on $\mathbb{F}$ such that

$$\mathrm{Aut}(\mathbb{F}) \subset \mathrm{Aut}(\mathbb{F}, <).$$

Proof.

Set of binary relations on $\mathbb{F}$: $2^{\mathbb{F}\times\mathbb{F}}$, compact.
$\mathrm{Aut}(\mathbb{F})$ acts continuously on $2^{\mathbb{F}\times\mathbb{F}}$: $gR(x,y)$ iff $R(g^{-1}x, g^{-1}y)$.
$LO(\mathbb{F}) \subset 2^{\mathbb{F}\times\mathbb{F}}$ closed, $\mathrm{Aut}(\mathbb{F})$-invariant.
Any $<$ fixed point is as required. □

So those countable ultrahomogeneous classes which have a chance to have extremely amenable automorphism groups are ordered.

The Ramsey property

Definition

Let $\mathbb{F}$ be a countable ultrahomogeneous structure.
Say $\mathbb{F}$ has the Ramsey property when for any:

- *$A \subset \mathbb{F}$ finite (small structure, to be colored),*
- *$B \subset \mathbb{F}$ finite (medium structure, to be reconstituted),*
- *$k \in \mathbb{N}$ (number of colors),*

...the following holds:

Whenever copies of A in $\mathbb{F}$ are colored with k colors,
there is $\tilde{B} \cong B$ where all copies of B have same color.

Remark

In fact, it is a property of $\mathrm{Age}(\mathbb{F})$ only.

Extreme amenability and Ramsey property

Theorem (Kechris-Pestov-Todorcevic, 05)

Let $\mathbb{F}$ be an ordered countable ultrahomogeneous structure. TFAE:

i) $\mathrm{Aut}(\mathbb{F})$ *is extremely amenable.*

ii) $\mathbb{F}$ *has the Ramsey property.*

Definition

If $A, B \subset \mathbb{F}$, $\binom{B}{A}$ denotes the set of all substructures of B isomorphic to A (copies of A in B).

Extreme amenability implies Ramsey property

Assume $G := \mathrm{Aut}(\mathbb{F})$ is extremely amenable.
Let $k \in \mathbb{N}$, $A, B \subset \mathbb{F}$, finite, $\chi : \binom{\mathbb{F}}{A} \longrightarrow k$.
Space of k-colorings of $\binom{\mathbb{F}}{A}$: $k^{\binom{\mathbb{F}}{A}}$, compact.
G acts on $k^{\binom{\mathbb{F}}{A}}$ continuously:

$$g \cdot c : \tilde{A} \mapsto c(g^{-1}(\tilde{A})).$$

$\overline{G \cdot \chi} \subset k^{\binom{\mathbb{F}}{A}}$ compact, G-invariant.
So there is $c \in \overline{G \cdot \chi}$, G-fixed point. Note that c is constant.
Since $c \in \overline{G \cdot \chi}$,

$$\exists g \in G \quad c \restriction \binom{B}{A} = g \cdot \chi \restriction \binom{B}{A}.$$

So $g \cdot \chi$ constant on $\binom{B}{A}$, ie g constant on $\binom{g^{-1}(B)}{A}$. $\square$

Ramsey property implies extreme amenability

Assume $\mathbb{F}$ has the Ramsey property.
Write $\mathbb{F} = \{x_n : n \in \mathbb{N}\}$, $A_m = \{x_n : n \leq m\}$
$\mathbb{F}$ is ordered, so

$$\text{setwise stabilizer of } A_m = \text{pointwise stabilizer of } A_m.$$

$G/Stab(A_m) \cong \binom{\mathbb{F}}{A_m}$
Recall: Left-invariant metric on $G = \mathrm{Aut}(\mathbb{F})$

$$d(g,h) = 1/2^n, \text{ with } n = \min\{k \in \mathbb{N} : g(k) \neq h(k)\}.$$

Elements of $G/Stab(A_m)$ have diameter $< 1/2^m$.

Proposition

Let $k \in \mathbb{N}$, $m \in \mathbb{N}$, $F \subset G$ finite.
Let $\bar{f} : G \longrightarrow k$ constant on elements of $G/Stab(A_m)$.
Then there is $g \in G$ so that $\bar{f}$ is constant on gF.

Proof.

$\bar{f}$ induces $f : G/Stab(A_m) \longrightarrow k$, ie $f : \binom{\mathbb{F}}{A_m} \longrightarrow k$.
$\{[h] : h \in F\}$ is a finite family of substructures of $\mathbb{F}$, all isomorphic to A_m.
Fix B large enough so that

$$\{[h] : h \in F\} \subset \binom{B}{A_m}$$

By Ramsey property, find $\tilde{B} \cong B$, f constant on $\binom{\tilde{B}}{A_m}$, with value $i < k$.
Because $\mathbb{F}$ is ordered and ultrahomogeneous: $\exists g \in G \quad g''B = \tilde{B}$.
Then $\bar{f}$ is constant on gF:
If $h \in F$, then $\bar{f}(gh) = f([gh]) = f(g''[h]) = i$ because $g''[h] \in \binom{\tilde{B}}{A_m}$. □

Proposition

Let $p > 0$, $f : G \longrightarrow \mathbb{R}^p$ *uniformly continuous, bounded,* $F \subset G$ *finite,* $\varepsilon > 0$. *Then*

$$\exists g \in G \;\; \forall h, h' \in F \;\; |f(gh) - f(gh')| < \varepsilon$$

Proof.

As subsets of G, elements of $G/Stab(A_m)$ have diameter $< 1/2^m$.
So can find m, $\bar{f} : G \longrightarrow \mathbb{R}^p$ constant on elements of $G/Stab(A_m)$ so that

$$\|f - \bar{f}\|_\infty < \varepsilon$$

By previous proposition, find $g \in G$ so that $\bar{f}$ constant on gF.
Then f is ε-constant on gF.

□

Proposition

G is extremely amenable.

Proof.

Let G act continuously on K compact.
For $p > 0$, $f : G \longrightarrow \mathbb{R}^p$ uniformly continuous, bounded, $F \subset G$ finite, $\varepsilon > 0$, set

$$E_{f,\varepsilon,F} = \{x \in X : \forall g \in F \|f(x) - f(gx)\| \leq \varepsilon\}$$

$(E_{f,\varepsilon,F})_{f,\varepsilon,F}$ family of closed subsets of K with finite intersection property.
Fix

$$x_0 \in \bigcap_{f,\varepsilon,F} E_{f,\varepsilon,F}$$

Then x_0 is fixed by G:
If not, find $g_0 \in G$ such that $gx_0 \neq x_0$.
Find $f_0 : X \longrightarrow [0,1]$ uniformly continuous with $f_0(x_0) = 0$, $f_0(g_0x_0) = 1$.
Then $x_0 \notin E_{f_0,1,\{g_0\}}$, a contradiction. □

The very first example

- Finite linear orders:

Theorem (Ramsey, 30)

$\mathcal{LO}$ has the Ramsey property.

Corollary (Pestov, 98)

$\mathrm{Aut}(\mathbb{Q},<)$ *extremely amenable.*

Corollary (Pestov, 98)

$Homeo_+(\mathbb{R})$ (pointwise convergence topology) extremely amenable.

Proof.

$\mathrm{Aut}(\mathbb{Q},<) \hookrightarrow Homeo_+(\mathbb{R})$ densely. □

Example: Metric spaces

• Finite ordered metric spaces with rational distances: $\mathbb{U}_{\mathbb{Q}}^{<} = (\mathbb{U}_{\mathbb{Q}}, <^{\mathbb{U}_{\mathbb{Q}}})$.

Theorem (Nešetřil, 05)

$\mathcal{M}_{\mathbb{Q}}^{<}$ *has the Ramsey property.*

Corollary

$\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}}, <^{\mathbb{U}_{\mathbb{Q}}})$ *extremely amenable.*

Corollary

$\mathrm{iso}(\mathbb{U})$ *extremely amenable.*

Proof.

$\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}}, <^{\mathbb{U}_{\mathbb{Q}}}) \hookrightarrow \mathrm{iso}(\mathbb{U})$ densely. □