Structural Ramsey theory and topological dynamics I

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Part I

Outline

Outline and goals

Describe and elaborate on a theory developed in 2005 by Kechris, Pestov and Todorcevic.

- Main references:
  - A. Kechris, V. Pestov and S. Todorcevic, Fraïssé limits, Ramsey theory and topological dynamics of automorphism groups, GAFA, 15, p105–189, 2005.
  - V. Pestov, Dynamics of infinite dimensional groups - The Ramsey-Dvoretzky-Milman phenomenon, University Lecture Series, 40, 2006.
- Outline
  - History of extremely amenable groups.
  - Closed subgroups of S_∞, Fraïssé classes and Fraïssé limits.
  - Extreme amenability and Ramsey property.
  - Universal minimal flows.
  - Perspectives.

Part II

Extreme amenability

Brouwer's theorem and its generalizations

Theorem (Brouwer, 1911)

Let $n \in \mathbb{N}$, $n > 0$ *and* $f : [0,1]^n \longrightarrow [0,1]^n$ *continuous.*
Then f *has a fixed point.*

Theorem (Schauder, 1930)

Idem with $[0,1]^n$ *replaced by a compact convex subset of a normed vector space.*

Theorem (Tychonoff, 1935)

Idem if normed space replaced by locally convex topological vector space.

Question

What if, instead of one single $f : K \longrightarrow K$, *we have several maps?*

Markov-Kakutani theorem

Theorem (Markov, 1936 - Kakutani, 1938)

Let K be compact convex subset of a locally convex topological vector space,
F a semigroup of affine, continuous, commuting functions $K \longrightarrow K$.
Then there is a common fixed point for all elements of F

Question

What semigroups have the same property?

Amenability and Day's theorem

Definition

Let S be a semigroup, $\ell^\infty(S)$ Banach space of bounded functions $S \longrightarrow \mathbb{R}$, sup norm. A mean on $\ell^\infty(S)$ is an element μ of $\ell^\infty(S)^*$ such that:

- $\mu(f) \geq 0$ *whenever* $f \geq 0$.
- $\mu(1) = 1$

For $f \in \ell^\infty(S)$ and $s \in S$, set $sf : x \mapsto f(sx)$.
μ is left invariant when $\mu(sf) = \mu(f)$ for all f, s.
S is amenable when there is a left invariant mean on $\ell^\infty(S)$.

Theorem (Day, 61)

Conclusion of Markov-Kakutani theorem holds when S is an amenable semigroup of continuous affine maps $K \longrightarrow K$.

Extreme amenability and Mitchell's theorem

Definition

A semigroup S is extremely amenable when there is a left invariant multiplicative mean on $\ell^\infty(S)$, ie $\mu(fg) = \mu(f)\mu(g)$ for all f, g.

Theorem (Mitchell, 66)

Let S be a semigroup. Then S is extremely amenable iff every action of S by continuous functions on any compact space has a fixed point.

Definition

A topological semigroup S is amenable (resp. extremely amenable) when there is left invariant (resp. left invariant and multiplicative) mean on $\ell^\infty(S) \cap \mathcal{C}(S, \mathbb{R})$.

The case of topological semigroups

Theorem

Let S be a topological semigroup. TFAE:

1. *S is amenable.*
2. *Every continuous action of S on a compact convex subset of a locally convex topological vector space has a fixed point.*
3. *Every continuous action of S on a compact space has an invariant Borel probability measure.*

Theorem

Let S be a topological semigroup. TFAE:

1. *S is extremely amenable.*
2. *Every continuous action of S of a compact space has a fixed point.*

Are there extremely amenable groups?

Remark

Extremely amenable groups mentioned in 67 by Granirer as hypothetical objects. Question of their existence appears in print in 70 (Mitchell).

Theorem (Herrer-Christensen, 75)

There is a Polish Abelian extremely amenable group.

Theorem (Veech, 77)

Let G be non-trivial and locally compact.
Then G is not extremely amenable.

Extremely amenable groups: examples everywhere!

Examples

1. *$O(\ell_2)$, pointwise convergence topology (Gromov-Milman, 84).*
2. *Measurable maps $[0,1] \to \mathbb{S}^1$ (Furstenberg-Weiss, unpub-Glasner, 98)*

$$d(f,g) = \int_0^1 d(f(x), g(x)) d\mu.$$

3. $\mathrm{Aut}(\mathbb{Q}, <)$, *product topology induced by $\mathbb{Q}^{\mathbb{Q}}$ (Pestov, 98).*
4. *$Homeo_+([0,1])$, $Homeo_+(\mathbb{R})$, pointwise convergence topology (Pestov, 98).*
5. $\mathrm{iso}(\mathbb{U})$, *pointwise convergence topology, $\mathbb{U}$ the Urysohn metric space (Pestov, 02).*

Remark

Examples 3, 4, and 5 by Pestov use some Ramsey theoretic results.

The work of Kechris, Pestov and Todorcevic, I

Definition

S_∞: the group of permutations of $\mathbb{N}$.
Left invariant metric:

$$d(g,h) = 1/2^n, \text{ with } n = \min\{k \in \mathbb{N} : g(k) \neq h(k)\}.$$

This topology is Polish (separable, metrizable with a complete metric) but not locally compact.

Theorem (Kechris - Pestov - Todorcevic, 05)

There is a link between extreme amenability and Ramsey theory when G is a closed subgroup of S_∞.

Part III

Closed subgroups of S_∞

Ultrahomogeneous structures

Definition

Let $L = \{R_i : i \in I\}$ *be a countable, first order, relational language.*
An L*-structure* $\mathbb{F}$ *is* ***ultrahomogeneous*** *when every isomorphism between finite substructures of* $\mathbb{F}$ *extends to an automorphism of* $\mathbb{F}$*.*

Example

$L = \{<\}$, $<$ *binary relation symbol.*
$\mathbb{F} = (\mathbb{Q}, <)$.

More examples later.

Closed subgroups of S_∞ and countable ultrahomogeneous structures

Proposition

- *If* $\mathbb{F}$ *is countable (WLOG,* $\mathbb{F} = (\mathbb{N}, \ldots)$*),*
  then $\mathrm{Aut}(\mathbb{F})$ *is a closed subgroup of* S_∞*.*
- *If* G *closed subgroup of* S_∞*, then there is*
  - L *countable language,*
  - $\mathbb{F}_G = (\mathbb{N}, \ldots)$ *countable ultrahomogeneous* L*-structure*

  such that

$$G = \mathrm{Aut}(\mathbb{F}_G).$$

  Relations of arity n*: orbits of* $G \curvearrowright \mathbb{N}^n$*.*

Corollary

The closed subgroups of S_∞ *are exactly the automorphism groups of countable ultrahomogeneous structures.*

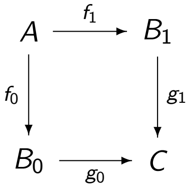
Combinatorial properties of classes of finite structures

L a countable first order relational language, $\mathcal{K}$ a class of finite L-structures.

Definition

$\mathcal{K}$ satisfies:

1. *hereditarity when it is is closed under substructures.*
2. *amalgamation when for all $A, B_i \in \mathcal{K}$ ($i = 0, 1$), embeddings $f_i : A \longrightarrow B_i$, there is C and embeddings $g_i : B_i \longrightarrow C$ such that $g_0 \circ f_0 = g_1 \circ f_1$.*



3. *joint embedding property: for all $A, B \in \mathcal{K}$, there is $C \in \mathcal{K}$ such that A, B embed in C.*

Fraïssé classes

Definition

$\mathcal{K}$ is a Fraïssé class when it is countable, has elements of arbitrary high cardinality, and satisfies properties 1, 2 and 3.

Examples

- $\mathcal{LO}$ *finite linear orders,* $L = \{<\}$.
- $\mathcal{G}$ *finite graphs,* $L = \{E\}$ *adjacency relation symbol.*
- $\mathcal{M}_{\mathbb{Q}\cap[0,1]}$ *finite metric spaces with rational distances,* $L = \{d_q : q \in \mathbb{Q}\}$ *binary relational language,* $d_q^X(x, y)$ *when* $d^X(x, y) < q$.

Fraïssé's theorem

Proposition

Let $\mathbb{F}$ be a countable ultrahomogeneous L-structure.
$Age(\mathbb{F})$ the class of all finite substructures of $\mathbb{F}$.
Then $Age(\mathbb{F})$ is a Fraïssé class.

Theorem (Fraïssé, 54)

Let $\mathcal{K}$ be a Fraïssé class in some language countable L.
Then up to isomorphism, there is a unique countable ultrahomogeneous L-structure $\mathbb{F}$ for which

$$Age(\mathbb{F}) = \mathcal{K}.$$

Notation: $\mathbb{F} = Flim(\mathcal{K})$.

Fraïssé classes of graphs

Fraïssé classes of graphs classified by Lachlan-Woodrow, 80.

Examples

- *$\mathcal{CG}$ finite complete graphs:* $Flim(\mathcal{CG}) = K_\omega$.
  The countable infinite complete graph.
- *$\mathcal{G}$ finite graphs:* $Flim(\mathcal{G}) = \mathcal{R}$.
  The Rado graph, universal for countable graphs.
- *$\mathcal{G}_n$ K_n-free finite graphs:* $Flim(\mathcal{G}_n) = H_n$.
  Henson graphs, universal for countable K_n-free graphs.

Fraïssé classes of oriented graphs

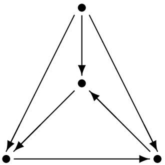
Fraïssé classes of oriented graphs classified by Cherlin, 98.

Examples

- $\mathcal{LO}$ *finite linear orders:* *Flim*$(\mathcal{LO}) = (\mathbb{Q}, <)$.
- $\mathcal{PO}$ *finite partial orders:* *Flim*$(\mathcal{PO}) = \mathbb{P}$.
  The countable ultrahomogeneous poset, universal for all countable posets.

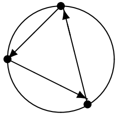
Oriented graphs, cont'd

- $\mathcal{C}$ finite local orders:
  Finite tournaments not embedding



$$Flim(\mathcal{C}) = S(2).$$

Vertices: Rational points of $\mathbb{S}^1$ (no antipodal pair).
Arcs: $x \to y$ iff (counterclockwise angle from x to y) $< \pi$.



Metric spaces

Fraïssé classes of finite metric spaces still not classified.

Examples

- $\mathcal{M}_S$ *finite metric spaces with distances in* S *(conditions on* S *needed, see Delhommé-Laflamme-Pouzet-Sauer):*

  $$Flim(\mathcal{M}_S) = \mathbb{U}_S.$$

  The countable Urysohn space with distances in S*, universal for countable metric spaces with distances in* S*.*

  - *Interesting cases: finite,* $\mathbb{Q}$*,* $\mathbb{N}$*.*

- $\mathcal{U}$ *finite ultrametric spaces with distances in* $\{1/2^n : n \in \mathbb{N}\}$*:*

  $$\forall x, y, z \;\; d(x,z) \leq \max(d(x,y), d(y,z)).$$

  $$Flim(\mathcal{U}) = \mathbb{U}^{ult}.$$

  Dense subspace of the Baire space $\mathbb{N}^{\mathbb{N}}$ *(eventually* 0 *sequences).*

Structures with operations

The following are not relational, but can be coded as relational structures, and then correspond to cofinal subclasses of relational Fraïssé classes.

Examples

- $\mathcal{BA}$ *finite Boolean algebras,* $L = \{0, 1, -, \wedge, \vee\}$*:*

  $$Flim(\mathcal{BA}) = B_\infty.$$

  The countable atomless Boolean algebra, universal for countable Boolean algebras.

- $\mathcal{V}_F$ *finite vector spaces,* F *finite field,* $L = \{+\} \cup \{f_\alpha : \alpha \in F\}$*:*

  $$Flim(\mathcal{V}_F) = F^{<\omega}.$$

  The countable infinite dimensional vector space over F*.*

Summary

- Some Polish, non locally compact groups G are extremely amenable: Every continuous action of G on a compact space has a fixed point.
- When G closed subgroup of S_∞, extreme amenability will have a finite combinatorial characterization. This is so because:
- $G = \mathrm{Aut}(\mathbb{F})$, $\mathbb{F}$ a countable ultrahomogeneous first order structure.
- $\mathbb{F}$ is the Fraïssé limit of a class $\mathcal{K}$ of finite structures.
- G will be extremely amenable iff some combinatorial phenomenon takes place at the level of $\mathcal{K}$ (Ramsey type properties).