Uncountable homogeneous structures

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The setup

Definition

- We fix a Fraïssé class $\mathcal{F}$ of finitely generated structures in a countable language.
- We denote by $\sigma\mathcal{F}$ the class of all unions of countable chains in $\mathcal{F}$, in other words, the class of all countable structures whose age is in $\mathcal{F}$.
- We denote by $\overline{\mathcal{F}}$ the class of all structures whose age is in $\mathcal{F}$.
- We denote by $\mathbb{U}$ the Fraïssé limit of $\mathcal{F}$.

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What can we say about uncountable homogeneous structures in $\overline{\mathcal{F}}$?
Do they always exist?

An example from basic group theory

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Note that, up to isomorphism, $\mathcal{F} = \{1, \langle \mathbb{Z}, + \rangle\}$.
Note that $\sigma\mathcal{F}$ consists of all locally cyclic torsion-free groups. All of them are isomorphic to subgroups of the Fraïssé limit $\mathbb{U} = \langle \mathbb{Q}, + \rangle$.

Theorem

Assume CH. If $\sigma\mathcal{F}$ has the amalgamation property then there exists a unique structure $\mathbb{U}_{\omega_1} \in \overline{\mathcal{F}}$ of cardinality $\aleph_1$ that is $\sigma\mathcal{F}$-homogeneous and universal.
Furthermore, every structure in $\overline{\mathcal{F}}$ of cardinality $\leq 2^{\aleph_0}$ embeds into $\mathbb{U}_{\omega_1}$.

Fraïssé-Jónsson theory

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Remark

This may fail without CH.

Ultrapowers

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Theorem (cf. Keisler 1964)

Assume the language of $\mathcal{F}$ is finite and relational and fix a non-principle ultrafilter p over ω. Then the ultrapower $\mathbb{U}^\omega/p$ is homogeneous and its age is $\mathcal{F}$.

Katětov functors

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Theorem (cf. Kubiś & Mašulović 2017)

Assume there exists a functor $K \colon \sigma\mathcal{F} \to \sigma\mathcal{F}$ *together with a natural transformation* η *from the identity to* K *such that* $K(\mathbb{U}) \approx \mathbb{U}$ *and* $\eta_{\mathbb{U}} \colon \mathbb{U} \to K(\mathbb{U})$ *is a nontrivial embedding.*

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Then $K^{\omega_1}(\mathbb{U})$ *is homogeneous.*

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Example

Let $\mathcal{F}$ consist of all pairs of the form $\langle F, f \rangle$, where $f \colon F \to \omega$ is a one-to-one function. Then $\mathbb{U} \approx \langle \omega, \mathrm{id}_\omega \rangle$ and $\mathcal{E}\mathbb{U} = \{\mathrm{id}_\mathbb{U}\}$.

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Theorem

Assume $\mathcal{M}$ is a sub-monoid of $\mathcal{E}\mathbb{U}$ containing $\mathrm{Aut}(\mathbb{U})$ as well as at least one nontrivial embedding. Furthermore, assume that $\mathcal{M}$ has the amalgamation property.

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Then there exists an uncountable homogeneous structure with age $\mathcal{F}$.

Remark

The assumptions above are fulfilled if the language of $\mathcal{F}$ is finite and relational, namely, in that case $\mathcal{E}\mathbb{U}$ has the amalgamation property.

Question

Does there exist a relational Fraïssé class $\mathcal{F}$ such that no uncountable structure with age $\mathcal{F}$ is homogeneous?

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There exists a relational Fraïssé classes $\mathcal{F}$ such that no $X \in \sigma\mathcal{F}$ is an amalgamation base.

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Does there exist a relational Fraïssé class $\mathcal{F}$ such that no uncountable structure with age $\mathcal{F}$ is homogeneous?

Claim

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Proof.

One of the examples is the class of all finite anti-metric spaces with natural values. □

THE END