Eggleston meets Mycielski – category case

Marcin Michalski, Robert Rałowski, Szymon Żeberski

Wrocław University of Science and Technology

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Theorem (Eggleston)

For every conull set $F \subseteq [0,1]^2$ there are a perfect set $P \subseteq [0,1]$ and conull $B \subseteq [0,1]$ such that $P \times B \subseteq F$.

Let $\Delta = \{(x,x) : x \in [0,1]\}$.

Theorem (Mycielski)

For every comeager or conull set $X \subseteq [0,1]^2$ there exists a perfect set $P \subseteq [0,1]$ such that $P \times P \subseteq X \cup \Delta$.

H. G. Eggleston, Two measure properties of Cartesian product sets, The Quarterly Journal of Mathematics 5 (1954) 108-–115.

Mycielski J., Algebraic independence and measure, Fundamenta Mathematicae 61 (1967) 165–169.

Consider the Cantor space 2^ω and let T be a tree on ω, i.e. for each $\sigma \in T$ we have $\sigma \restriction n \in T$ for all natural n.

A body of a tree T is the set

$$[T] = \{x \in 2^\omega : (\forall n \in \omega)(x \restriction n \in T)\}$$

of all infinite branches of T.

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The goal: to switch from $[0,1]^2$ to $2^\omega \times 2^\omega$, replace a perfect set with a body of some tree, and prove Egglestone Theorem or its mixture with Mycielski Theorem for such a setting for the category.

Definition

We call a tree $T \subseteq 2^{<\omega}$

- a perfect or Sacks tree, if for each $\sigma \in T$ there is $\tau \in T$ such that $\sigma \subseteq \tau$ and $\tau^\frown 0, \tau^\frown 1 \in T$;
- uniformly perfect, if it is perfect and for all $\sigma, \tau \in T$ if $|\sigma| = |\tau|$ then $\sigma^\frown 0, \sigma^\frown 1 \in T \Leftrightarrow \tau^\frown 0, \tau^\frown 1 \in T$;
- a Silver tree, if it is perfect and for all $\sigma, \tau \in T$ with $|\sigma| = |\tau|$ we have $\sigma^\frown i \in T \Leftrightarrow \tau^\frown i \in T$ for $i = 1, 2$. Equivalently: if there is $x \in 2^\omega$ and an infinite set $A \subseteq \omega$ such that
$$(\forall \sigma \in T)(\forall n \in \mathrm{dom}(\sigma))(n \notin A \to \sigma(n) = x(n))$$
- a Spinas tree if for every $\sigma \in T$ there is $N_\sigma \in \omega$ such that for each $n \geq N_\sigma$ there are $\tau_0, \tau_1 \in T \cap 2^n$ such that $\sigma \subseteq \tau_0{}^\frown 0 \in T$ and $\sigma \subseteq \tau_1{}^\frown 1 \in T$;

Definition

A tree $T \subseteq \omega^{<\omega}$ is a Miller tree if for every $\sigma \in T$ there is $\tau \in T$ and infinite set $A \subseteq \omega$ such that $\sigma \subseteq \tau$ and $\tau^\frown n \in T$ for every $n \in A$.

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Theorem

There exists a dense G_δ set $G \subseteq \omega^\omega \times \omega^\omega$ such that $[T_1] \times [T_2] \not\subseteq G \cup \Delta$ for any Miller trees T_1 and T_2.

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Corollary

Category variant of Eggleston Theorem for a body of Miller tree does not hold.

Theorem

For every comeager set $G \subseteq (2^\omega \times 2^\omega)$ *there are a Silver tree* $T \subseteq 2^\omega$ *and a dense* G_δ*-set* $B \subseteq 2^\omega$ *such that* $[T] \times B \subseteq G$.

sketch of the proof.

$G = \bigcap_{n\in\omega} U_n$, U_n open and dense.
Fix a topological base $\{B_n : n \in \omega\}$.
Construct inductively sequences $\tau_n \in 2^{<\omega}$ and open V_n such that for all n

$$V_n \subseteq B_n;$$
$$[\tau_0{}^\frown i_0{}^\frown \tau_1{}^\frown i_1{}^\frown \ldots {}^\frown \tau_{n-1}{}^\frown i_{n-1}{}^\frown \tau_n] \times V_n \subseteq U_n$$

for every $(i_0, i_1, i_2, \ldots, i_{n-1}) \in 2^n$.
Set

$$t = \tau_0{}^\frown 0^\frown \tau_1{}^\frown 0^\frown \tau_2{}^\frown 0^\frown \tau_3{}^\frown \ldots,$$
$$A = \{|\tau_0|, |\tau_0| + |\tau_1| + 1, |\tau_0| + |\tau_1| + |\tau_2| + 2, \ldots\}.$$

Then $\{x \in 2^\omega : (\forall n \notin A)(x(n) = t(n))\}$ is a body of some Silver tree T and $B = \bigcap_{n\in\omega} \bigcup_{m\geq n} V_m$ is the desired dense G_δ set. □

Theorem

For every comeager $G \subseteq (2^\omega \times 2^\omega)$ *there are a Spinas tree* $T \subseteq 2^{<\omega}$ *and a dense* G_δ*-set* $B \subseteq 2^\omega$ *such that* $[T] \times B \subseteq G$. *Moreover* T *contains a Silver tree.*

Remark

There exists an open dense set $U \subseteq 2^\omega \times 2^\omega$ *such that* $[T] \times [T] \not\subseteq U \cup \Delta$ *for any Silver tree* T. *Thus we cannot have* $[T] \subseteq B$ *in previous theorems.*

Theorem

Let $G \subseteq 2^\omega \times 2^\omega$ *be comeager. Then there exist a uniformly perfect tree* $T \subseteq 2^{<\omega}$ *and a dense* G_δ *set* $B \subseteq 2^\omega$ *such that* $[T] \subseteq B$ *and* $[T] \times B \subseteq G \cup \Delta$.

Theorem

Let $G \subseteq 2^\omega \times 2^\omega$ be comeager. Then there exist a uniformly perfect tree $T \subseteq 2^{<\omega}$ and a dense G_δ set $B \subseteq 2^\omega$ such that $[T] \subseteq B$ and $[T] \times B \subseteq G \cup \Delta$.

Problem

Does every comeager set $G \subseteq 2^\omega \times 2^\omega$ contain $([T] \times D) \setminus \Delta$, where $T \subseteq 2^{<\omega}$ is a Spinas tree and $D \subseteq 2^\omega$ is a dense G_δ set such that $[T] \subseteq D$?

Thank you!

Michalski M., Rałowski R. Żeberski Sz., Around Eggleston theorem, arXiv:2307.07020.

M. Michalski, R. Rałowski, and Sz. Żeberski, Mycielski among trees, Mathematical Logic Quarterly 67 (2021) 271-–281.

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