Few words on P-measures

almost σ-additive measures on ω

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joint work with Piotr Borodulin-Nadzieja and

Jonathan Cancino Manríquez

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Preliminaries

- Ultrafilters - non-principal, on ω.
- Measures - finitelly additive, vanishing on points probability measures on ω.

Preliminaries

P-point

An ultrafilter $\mathcal{U}$ s.t. for every $\{A_n : n \in \omega\} \subset \mathcal{U}$ there is $A \in \mathcal{U}$ - a pseudointersection of $\{A_n : n\}$ i.e.

$$\forall_n \, A \subseteq^* A_n$$

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P-measure

A measure μ s.t. for every $\subseteq$-decreasing sequence $\langle A_n : n \in \omega\rangle$ there is a pseudointersection A such that

$$\mu(A) = \inf_n \mu(A_n)$$

In literature these are called measures with AP - Additive Property.

Basic Examples

Dirac Delta

$$\delta_{\mathcal{U}}(A) = \begin{cases} 1, & A \in \mathcal{U} \\ 0, & \text{else} \end{cases}$$

is a P-measure iff $\mathcal{U}$ is a P-point.

Density

$$d_{\mathcal{U}}(A) = \lim_{n \to \mathcal{U}} \frac{|A \cap n|}{n}$$

is an atomless P-measure if $\mathcal{U}$ is a P-point.

Existence

Fact

CH $\implies$ *there is a P-point.*

$\exists$ *P-point* $\implies$ $\exists$ *P-measure*

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Consistently there are neither P-points nor P-measures.

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Theorem (Shelah [Wim82], Mekler [Mek84])

Consistently there are neither P-points nor P-measures.

Theorem (Cancino Manríquez)

Consistently there is a P-measure but no P-points.

Silver Model

Definition

The Silver forcing is the set

$$\mathbb{S}_{\mathbb{I}} = \{f : A \to 2 : A \subseteq \omega \text{ is co-infinite}\}$$

ordered by reverse inclusion.

The Silver Model is obtained by forcing with countable support ω_2-iteration of $\mathbb{S}_{\mathbb{I}}$.

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But are there P-measures there?

Rudin-Blass ordering

Definition

$\mu \leqslant_{RB} \nu \iff$ there is a finite-to-one $f \in \omega^\omega$ s.t.

$$\forall_{A \subset \omega} \quad \mu(A) = \nu(f^{-1}[A])$$

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Fact

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For $V < W$ (models of ZFC) and $\mu, \nu \in V$ if $\mu \leqslant_{RB} \nu$ and $\nu' \in W$ is a P-measure extending ν then there is $\mu' \in W$ - a P-measure extending μ.

$$\begin{array}{ccc} \mu & \leqslant_{RB} & \nu \\ \downarrow & & \downarrow \\ \mu' & \leqslant_{RB} & \nu' \end{array}$$

Silver Model?

Theorem ?

No ultrafilter can be extended to a P-measure

Extended where?

Silver extensions

Theorem (Borodulin-Nadzieja, Cancino Manríquez, M.)

No ultrafilter from V can be extended to a P-measure in $V^{\mathbb{S}_{\mathbb{I}}^{\omega}}$ - a model obtained from V by forcing with ω **product** of $\mathbb{S}_{\mathbb{I}}$.

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Corollary

If $\mu \geqslant_{RB} \mathcal{U}$ (for some $\mathcal{U}$ - ultrafilter) then μ cannot be extended to a P-measure in $V^{\mathbb{S}_{\mathbb{I}}^{\omega}}$.

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Fact

For any $\mathcal{U}$ there is $\mathcal{V}$ s.t. $\mathcal{V} \leqslant_{RB} d_{\mathcal{U}}$.

Another Construction

Theorem (Sikorski, Kunen [Kun76])

Assume CH. There is a surjective boolean homomorphism

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Lambda

$$\lambda_\omega(A) = \lambda(\varphi(A))$$

is a P-measure.

Properties

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Every measure extending asymptotic density (so also $d_{\mathcal{U}}$) is of Maharam type $\mathfrak{c}$, while λ_ω is of Maharam type $\aleph_0$.

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Proposition [CH]

We can strengthen the construction of φ so that for each finite-to-1 $f \in \omega^\omega$ there is $N \subseteq \omega$ s.t.

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We can strengthen the construction of φ so that for each finite-to-1 $f \in \omega^\omega$ there is $N \subseteq \omega$ s.t.

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Then for every $\mathcal{U}$

$$\mathcal{U} \not\leq_{RB} \lambda_\omega.$$

Kill it

Definition

If for each $N \subseteq \omega$, $\varepsilon > 0$ and interval partition $\{I_n\}_n$ there is $\bar{c} \in \{-1, 1\}^\omega$ so that

$$\mu\left(\bigcup_n (N^{\bar{c}_n} \cap I_n)\right) < \varepsilon$$

then we say that μ is *Silver*.

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then we say that μ is *Silver*.

Theorem (Borodulin-Nadzieja, Cancino Manríquez, M.)

If μ is Silver then it cannot be extended to a P-measure in $V^{\mathbb{S}_{\mathbb{I}}^\omega}$.

Is there something else?

Fact

All ultrafilters are Silver.
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Question

Under CH, is there a measure that is not Rudin-Blass-above a Silver measure?

Thank You
for attention

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and attendance.

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