Few words on P-measures almost σ-additive measures on ω

Adam Morawski joint work with Piotr Borodulin-Nadzieja and Jonathan Cancino Manríquez

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Preliminaries

- Ultrafilters non-principal, on ω .
- Measures finitelly additive, vanishing on points probability measures on ω.

Preliminaries

P-point

An ultrafilter \mathcal{U} s.t. for every $\{A_n : n \in \omega\} \subset \mathcal{U}$ there is $A \in \mathcal{U}$ - a pseudointersection of $\{A_n : n\}$ i.e.

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P-measure

A measure μ s.t. for every \subseteq -decreasing sequence $\langle A_n : n \in \omega \rangle$ there is a pseudointersection *A* such that

$$\mu(A) = \inf_n \mu(A_n)$$

In literature these are called measures with AP - Additive Property.

Basic Examples

Dirac Delta

$$\delta_{\mathcal{U}}(A) = \begin{cases} 1, & A \in \mathcal{U} \\ 0, & \text{else} \end{cases}$$

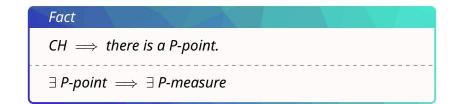
is a P-measure iff $\,\mathcal{U}$ is a P-point.

Density

$$d_{\mathcal{U}}(A) = \lim_{n \to \mathcal{U}} \frac{|A \cap n|}{n}$$

is an atomless P-measure if \mathcal{U} is a P-point

Existence



Existence



 $CH \implies there is a P-point.$

 $\exists P$ -point $\implies \exists P$ -measure

Theorem (Shelah [Wim82], Mekler [Mek84])

Consistently there are neither P-points nor P-measures.

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Theorem (Shelah [Wim82], Mekler [Mek84])

Consistently there are neither P-points nor P-measures.

Theorem (Cancino Manríquez)

Consistently there is a P-measure but no P-points.

Silver Model

Definition

The Silver forcing is the set

$$\mathbb{S}_{\mathbb{I}} = \{ f : A \to 2 : A \subseteq \omega \text{ is co-infinite} \}$$

ordered by reverse inclusion.

The Silver Model is obtained by forcing with countable support ω_2 -iteration of $\mathbb{S}_{\mathbb{I}}$.

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But are there P-measures there?

Rudin-Blass ordering

Definition

 $\mu \leqslant_{RB} \nu \iff$ there is a finite-to-one $f \in \omega^{\omega}$ s.t. $\forall_{A \subset \omega} \quad \mu(A) = \nu(f^{-1}[A])$

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If ν is a P-measure and $\mu \leq_{\textit{RB}} \nu$ then μ is a P-measure.

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For V < W (models of ZFC) and $\mu, \nu \in V$ if $\mu \leq_{RB} \nu$ and $\nu' \in W$ is a P-measure extending ν then there is $\mu' \in W$ - a Pmeasure extending μ .



Theorem ?

No ultrafilter can be extended to a P-measure

Extended where?

Theorem (Borodulin-Nadzieja, Cancino Manríquez, M.)

No ultrafilter from *V* can be extended to a P-measure in $V^{\mathbb{S}_{\mathbb{I}}^{\omega}}$ - a model obtained from *V* by forcing with ω **product** of $\mathbb{S}_{\mathbb{I}}$.

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Corollary

If $\mu \geq_{RB} \mathcal{U}$ (for some \mathcal{U} - ultrafilter) then μ cannot be extended to a P-measure in $V^{\mathbb{S}_1^{\omega}}$.

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Fact

For any \mathcal{U} there is \mathcal{V} s.t. $\mathcal{V} \leq_{RB} d_{\mathcal{U}}$.

Theorem (Sikorski, Kunen [Kun76])

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Theorem (Sikorski, Kunen [Kun76])

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and ker(φ) is a *P*-ideal.

Lambda

$$\lambda_{\omega}(\mathbf{A}) = \lambda(\varphi(\mathbf{A}))$$

is a P-measure.

Properties

Fact

Every measure extending asymptotic density (so also $d_{\mathcal{U}}$) is of Maharam type c, while λ_{ω} is of Maharam type \aleph_0 .

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Proposition [CH]

We can strengthen the construction of φ so that for each finite-to-1 $f \in \omega^{\omega}$ there is $N \subseteq \omega$ s.t.

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We can strengthen the construction of φ so that for each finite-to-1 $f \in \omega^{\omega}$ there is $N \subseteq \omega$ s.t.

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Then for every $\ensuremath{\mathcal{U}}$

 $\mathcal{U} \not\leq_{RB} \lambda_{\omega}.$

Definition

If for each $N \subseteq \omega$, $\varepsilon > 0$ and interval partition $\{I_n\}_n$ there is $\overline{c} \in \{-1, 1\}^{\omega}$ so that

$$u\left(\bigcup_n (N^{\overline{c}_n} \cap I_n)\right) < \varepsilon$$

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then we say that μ is *Silver*.

Theorem (Borodulin-Nadzieja, Cancino Manríquez, M.)

If μ is Silver then it cannot be extended to a P-measure in $V^{\mathbb{S}_{\mathbb{I}}^{\omega}}$.

Fact

All ultrafilters are Silver. Densities, λ_{ω} and all measures of countable Maharam type are Rudin-Blass-above some Silver measures.

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Question

Under CH, is there a measure that is not Rudin-Blassabove a Silver measure? Thank You for attention

Thank You for attention and attendance. David Chodounský and Osvaldo Guzmán. There are no P-points in Silver extensions. Israel J. Math., 232(2):759–773, 2019.

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