The Nikodym property and filters on ω

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Nikodym property of Boolean algebra

A Boolean algebra $\mathcal{A}$ has the *Nikodym property* if every pointwise null sequence of measures on $\mathcal{A}$ is uniformly bounded.

The Nikodym property - examples

Examples of Boolean algebras with the Nikodym property

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if $x_n \to x$, then consider the sequence of measures $\mu_n = n(\delta_{x_n} - \delta_x)$.

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Question

For which filters F on ω, if N_F embeds into the Stone space $St(\mathcal{A})$ of a Boolean algera $\mathcal{A}$, then $\mathcal{A}$ does not have the Nikodym property?

Main definition

A Borel measure μ on a topological space X is *finitely supported* if $\mu = \sum_{i=1}^{n} \alpha_i \delta_{x_i}$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and distinct $x_1, \ldots, x_n \in X$.

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$\mathcal{AN}$ - the class of all ideals on ω whose dual filters do not have the Nikodym property

Main motivation

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Theorem

If N_F homeomorphically embeds into $St(\mathcal{A})$ and F does not have the Nikodym property, then $\mathcal{A}$ does not have the Nikodym property.

Characterization in terms of non-negative measures

Proposition

If a filter F on ω does not have the Nikodym property, then there is an anti-Nikodym sequence of measures (μ_n) on N_F which is disjointly supported, that is, $supp(\mu_k) \cap supp(\mu_l) = \varnothing$ for every $k \neq l \in \omega$.

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A filter F on ω has the Nikodym property if and only if there is no sequence $\langle \mu_n : n \in \omega \rangle$ of finitely supported non-negative measures on N_F with disjoint supports such that:

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- $supp(\mu_n) \subseteq \omega$ *for every* $n \in \omega$,
- $\lim_{n\to\infty} \mu_n(\omega) = \infty$,
- $\lim_{n\to\infty} \mu_n(\omega \setminus A) = 0$ *for every* $A \in F$.

Submeasures on ω

Definition

$\varphi : \mathcal{P}(\omega) \to [0, \infty]$ is a *submeasure* if

- $\varphi(\emptyset) = 0$ and $\varphi(\{n\}) < \infty$ for every $n \in \omega$,
- $\varphi(X) \leq \varphi(Y)$ whenever $X \subseteq Y$,
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A submeasure φ on ω is *lower semicontinuous (lsc)* if $\varphi(A) = \lim_{n\to\infty} \varphi(A \cap [0,n])$ for every $A \subseteq \omega$. In particular, every non-negative measure μ on ω is an lsc submeasure.

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Exhaustive ideal

φ - lsc submeasure

The following ideal on ω is called *the exhaustive ideal* of φ:

$$Exh(\varphi) = \{A \subseteq \omega : \lim_{n\to\infty} \varphi(A \setminus [0,n]) = 0\}$$

Special types of submeasures

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Density submeasure

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An example is an *asymptotic density* on ω defined by:

$$\varphi_d(A) = \sup_{n \in \omega} \left| A \cap [2^n, 2^{n+1}) \right| / 2^n.$$

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- *F does not have the Nikodym property;*
- *there is a density submeasure φ on ω such that $\varphi(\omega) = \infty$ and $F \subseteq Exh(\varphi)^*$;*
- *there is a non-pathological lsc submeasure φ on ω such that $\varphi(\omega) = \infty$ and $F \subseteq Exh(\varphi)^*$.*

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Definition

Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on ω. We write that $\mathcal{I} \leq_K \mathcal{J}$ if there is a function $f: \omega \to \omega$ such that $f^{-1}[I] \in \mathcal{J}$ for all $I \in \mathcal{I}$.

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Grothendieck property of Boolean algebra

A Boolean algebra $\mathcal{A}$ has the *Grothendieck property* if the Banach space $C\big(St(\mathcal{A})\big)$ has the Grothendieck property.

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- (Talagrand, 1984) example under the Continuum Hypothesis of the algebra with (G) but without (N)

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- (Talagrand, 1984) example under the Continuum Hypothesis of the algebra with (G) but without (N)
- (Sobota & Zdomskyy, preprint 2023) example under the Martin's Axiom of the algebra with (G) but without (N)

Comparison with the BJN property

Theorem (Marciszewski, Sobota)

Let F be a filter on ω and $\mathcal{A}$ a Boolean algebra. Then,

- *if N_F has the BJN property and embeds into $St(\mathcal{A})$ then $\mathcal{A}$ does not have the Grothendieck property;*

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Theorem

For a density ideal $\mathcal{I}$ on ω we have $\mathcal{I} \in \mathcal{AN}$ if and only if $\mathcal{I} <_K \mathcal{Z}$.

The end

Thank You! :)