THEOREM (SHELAH). ASSUME $\mathfrak{s} < \mathfrak{a}$. THEN $RPC(\omega)$ HOLDS.

PROOF. LET $\mathfrak{D}$ BE AN ARBITRARY DENSE SUBSET OF $[\omega]^{\omega}$. WE WILL FIND A COMPLETELY SEPARABLE MAD FAMILY $\mathcal{A} \subseteq \mathfrak{D}$.

FIX A SPLITTING FAMILY $\{C_\alpha : \alpha < \mathfrak{s}\}$ AND DENOTE $C_\alpha^1 = C_\alpha$, $C_\alpha^0 = \omega \setminus C_\alpha$.

CONSIDER A TREE $T \subseteq {}^{<\mathfrak{s}}2$.

NOTATION:

$succ(T) = \{s \in T : dom(s)$ IS A SUCCESSOR ORDINAL$\}$,

$cl(T) = \{s \in {}^{<\mathfrak{s}}2 : \forall \alpha \in dom(s)\ s \restriction \alpha \in T\}$.

THE PROOF WILL GO BY INDUCTION TO 2^{ω}. WE SHALL BUILD A

A SEQUENCE OF PAIRS

$\langle T_\eta, \langle A_s : s \in succ(T_\eta) \rangle$ FOR $\eta < 2^\omega$ IN SUCH A WAY THAT $|T_\eta| \leq \omega + |\eta|$ AND FOR EACH $\eta < \xi < 2^\omega$, T_η IS A SUBTREE OF T_ξ.

WE DEMAND:

a) IF $s, t \in succ(T_\eta)$ ARE SUCH THAT $s \subsetneq t$, THEN $|A_s \cap A_t| < \omega$;

b) FOR EACH $s \in succ(T_\eta)$ AND FOR EACH $\alpha \in dom(s)$, $A_s \subseteq^* C_\alpha^{s(\alpha)}$.

GIVEN $s \in succ(T_\eta)$, A SET OF CANDIDATES WILL BE $\mathcal{X}_s =$

$$\{D \in \mathcal{D} : \forall \alpha \in dom(s) \; |D \cap A_{s \restriction \alpha}| < \omega \;\&\; D \subseteq^* C_\alpha^{s(\alpha)}\}$$

ENUMERATE $[\omega]^\omega = \langle M_\eta : \eta < 2^\omega \rangle$ IN SUCH A WAY THAT EACH SET APPEARS 2^ω- TIMES.

START WITH $T_0 = \emptyset$.

IF $\eta < 2^\omega$ IS A LIMIT ORDINAL, LET $T_\eta = \bigcup \{ T_\xi : \xi < \eta \}$.

SUCCESSOR STEP: WE KNOW $\langle T_\eta, \langle A_s : s \in succ(T_\eta) \rangle \rangle$ AND M_η.

IF THERE IS A FINITE SET $\{s_1, s_2, \cdots s_n\} \subseteq succ(T_\eta)$ SUCH THAT

$$M_\eta \subseteq^* A_{s_1} \cup A_{s_2} \cup \dots \cup A_{s_n},$$

THEN LET $T_{\eta+1} = T_\eta$.

IN THE OPPOSITE CASE WE NEED TO EXTEND T_η BY ADDING SOME s AND A_s WITH $A_s \subseteq M_\eta$.

FORGET THE SUBSCRIPT: $T = T_2$, $M = M_2$

DENOTE BY $\mathcal{J}$ THE IDEAL GENERATED BY $\{A_s : s \in succ(T)\}$.

LET US CONSIDER TWO SUBSETS OF T, DEPENDING ON M:

$F_M = F_M^1 \cup F_M^2$, WHERE

$$F_M^1 = \{s \in cl(T) : (\forall X \in \mathcal{J})\ [M \setminus X]^\omega \cap \mathcal{K}_s \neq \emptyset\},$$

$$F_M^2 = \{s \in cl(T) : \text{THE SET } \{t \in T : A_t \neq \emptyset \ \&\ s \subseteq t \ \&\ |A_t \cap M| = \omega\} \text{ IS INFINITE}\},$$

$$\mathcal{S}\ell_M = \{s \in T : s^\frown\langle 0\rangle \in F_M \ \&\ s^\frown\langle 1\rangle \in F_M\}.$$

THERE ARE SEVERAL CASES TO CONSIDER:

CASE (1): THERE IS SOME $s \in F_M$ BELONGING TO $cl(T) \setminus T$;

CASE (2): NOT CASE (1) AND $\mathcal{S}pl_M = \emptyset$;

CASE (3): NONE OF THE ABOVE.

IN CASE (1), THE s MUST BE IN F_M^1, ACCORDING TO THE DEFINITIONS OF $cl(T)$ AND OF F_M^2.

<u>SUBCASE 1A</u> : $dom(s)$ IS A SUCCESSOR ORDINAL.

PICK SOME $M' \in [M]^\omega \cap \mathcal{H}_s$, THEN PICK $A_s \in [M']^\omega$ SUCH THAT $M' \cap A_s$ IS INFINITE. THEN EXTEND T TO $T \cup \{s\}$.

SUBCASE 1B: $dom(s)$ IS A LIMIT ORDINAL.

LET $\alpha = dom(s)$. SINCE THERE IS AN INFINITE SET IN $\mathcal{K}_s$, $\alpha < \delta$.

FOR EACH $X \in \mathcal{J}$, WE KNOW THAT $[M \setminus X]^\omega \cap \mathcal{K}_s \neq \emptyset$. SINCE $C^0_\alpha \cup C^1_\alpha = \omega$, THERE MUST BE SOME $i \in \{0,1\}$ WITH

$$[(M \cap C^i_\alpha) \setminus X]^\omega \cap \mathcal{K}_s \neq \emptyset$$

WHENEVER $X \in \mathcal{J}$. AS BEFORE, LET $t = s^\frown\langle i\rangle$, $A_t \in [M \cap C^i_\alpha]^\omega$ BE SUCH THAT $A_t \in \mathcal{K}_s$ AND THERE IS SOME $M' \in \mathcal{K}_s$, $M' \setminus A_t$ INFINITE, $A_t \subseteq M' \subseteq M$. EXTEND T TO $T \cup \{s, t\}$.

SO, IN BOTH SUBCASES WE SUCCEEDED TO FIND SOME $A_s \subseteq M$.

CASE 2. SINCE $\mathcal{S}pl_M = \emptyset$, LET $\tilde{s} = \bigcup\{s : s \in F_M\}$.

SUBCASE 2A: $\tilde{s} \in F^1_M$.

SINCE WE ASSUME NOT CASE 1, WE HAVE $F_M \subseteq T$ AND SO $dom(\tilde{s}) < \delta$.

PUT $\alpha = dom(\tilde{s})$. DECOMPOSE

$M = M_0 \cup M_1 \cup M_2$, WHERE

$M_2 = M \cap A_{\tilde{s}}$, IF $A_{\tilde{s}}$ IS DEFINED, I.E, IF α IS A SUCCESSOR ORDINAL

$M_2 = \emptyset$ OTHERWISE;

$M_0 = (M \setminus M_2) \cap C^0_\alpha$;

$M_1 = (M \setminus M_2) \cap C^1_\alpha$.

SINCE $\tilde{s} \in F^1_M$, WE HAVE THAT

$(\forall X \in \mathcal{J})\ [M \setminus X]^\omega \cap \mathcal{H}_{\tilde{s}} \neq \emptyset$.

HENCE THE SAME MUST HOLD FOR ONE SET M_i. BUT $M_2 \subseteq A_{\tilde{s}}$ AND SO $[M_2 \setminus A_{\tilde{s}}]^{\omega} = \emptyset$. SINCE $\mathcal{S}pl_M = \emptyset$, IT CANNOT HOLD SIMULTANEOUSLY FOR BOTH 0,1.

SO THERE IS PRECISELY ONE $i \in \{0,1\}$ WITH THE PROPERTY THAT $(\forall X \in \mathcal{J})\ [M_i \setminus X]^{\omega} \cap \mathcal{H}_{\tilde{s}} \neq \emptyset$.

AS $M_i \cap A_{\tilde{s}} = \emptyset$, WE HAVE THAT $\tilde{s}^\frown\langle i\rangle \in F_{M_i}$; SINCE $M_i \subseteq M$, $F_{M_i} \subseteq F_M$. SO $\tilde{s}^\frown\langle i\rangle \in F_M$ WHICH CONTRADICTS THE DEFINITION OF $\tilde{s}$.

SO THE SUBCASE 2A NEVER HAPPENS.

SUBCASE 2B: $\tilde{s} \notin F^1_M$.

$\tilde{\alpha} = dom(\tilde{s})$ MUST BE A LIMIT ORDINAL IN THIS CASE; $\tilde{\alpha} \leq \mathfrak{s}$.

FOR EACH $\alpha < \tilde{\alpha}$ CONSIDER THE NODE $\tilde{s} \restriction \alpha ^\frown \langle 1 - \tilde{s}(\alpha) \rangle = t_\alpha$. SINCE $\mathcal{S}pl_M = \emptyset$, $t_\alpha \notin F_M$. IN PARTICULAR, $t_\alpha \notin F^1_M$. SO THERE IS A FINITE FAMILY $\mathcal{A}_\alpha \subseteq \{A_s : s \in succ(T)\}$ SUCH THAT $[M \setminus \bigcup \mathcal{A}_\alpha]^\omega \cap \mathcal{K}_{t_\alpha} = \emptyset$.

THEN THE FAMILY $\bigcup_{\alpha < \tilde{\alpha}} \mathcal{A}_\alpha \cup \{A_{\tilde{s} \restriction \alpha} : \alpha < \tilde{\alpha}\}$ IS OF SIZE $\leq |\tilde{\alpha}| \leq \mathfrak{b} < \mathfrak{s}$, SO ITS TRACE ON M CANNOT BE A MAD FAMILY ON M. SO THERE IS AN INFINITE $M' \subseteq M$ SUCH THAT $|M' \cap A| < \omega$ FOR ALL A BELONGING

TO $\bigcup_{\alpha<\tilde{\alpha}} \mathcal{A}_\alpha \cup \{A_{\tilde{x}\restriction\alpha} : \alpha < \tilde{\alpha}\}$.

CLEARLY, AS $M' \subseteq M$, WE HAVE ALSO $F^1_{M'} \subseteq F^1_M$ AND $F^2_{M'} \subseteq F^2_M$.

NOW, WE ALWAYS REACH A CONTRADICTION:

– IF THERE IS SOME $\alpha < \tilde{\alpha}$ SUCH THAT C_α SPLITS M'. FOR A MINIMAL α WITH THIS PROPERTY WE HAVE THAT FOR EACH $X \in \mathcal{J}$, $M' \cap X$ IS IS FINITE AND CONSEQUENTLY $\tilde{x}\restriction\alpha^\frown\langle 0\rangle \in F^1_{M'}$ AND $\tilde{x}\restriction\alpha^\frown\langle 1\rangle \in F^1_{M'}$, TOO. BUT THIS CONTRADICTS TO THE ASSUMPTION $\mathcal{S}pl_M = \emptyset$.

– IF NO $\alpha < \tilde{\alpha}$ WITH C_α SPLITTING M' EXISTS, THEN:

- IF $\tilde{\alpha} = \delta$, WE HAVE A CONTRADICTION THAT $\{C_\alpha : \alpha < \delta\}$ IS A SPLITTING FAMILY;
- IF $\tilde{\alpha} < \delta$, THEN EITHER

$$[M' \cap C^0_{\tilde{\alpha}}]^\omega \cap \mathcal{K}_{\tilde{s}^\frown\langle 0\rangle} \neq \emptyset, \text{ OR}$$

$$[M' \cap C^1_{\tilde{\alpha}}]^\omega \cap \mathcal{K}_{\tilde{s}^\frown\langle 1\rangle} \neq \emptyset.$$

IN BOTH CASES WE HAVE A CONTRADICTION WITH THE DEFINITION OF $\tilde{s}$.

SO CASE 2 NEVER HAPPENS.

CASE 3. NOTICE THAT "NOT CASE 1 FOR M" IMPLIES "NOT CASE 1 FOR M'" FOR EACH INFINITE $M' \subseteq M$. WE HAVE PROVED ALREADY

THAT CASE (2) NEVER HAPPENS.

SO HAVING NOT CASE (2) WE KNOW THAT FOR EACH INFINITE $M' \subseteq M$, $\mathcal{S}pl_{M'} \neq \emptyset$.

LET US DEFINE FOR EACH $n \in \omega$ AND FOR EACH $\rho \in {}^{n}2$ A MAPPING $s(\rho) \in \mathcal{S}pl_M$ BY INDUCTION AS FOLLOWS:

$s(\emptyset)$ IS THE UNIQUE ELEMENT OF $\mathcal{S}pl_M$ WITH THE MINIMAL *dom*.

SUPPOSE $s(\rho)$ IS KNOWN, $i \in \{0,1\}$. LET $s(\rho^\frown\langle i\rangle)$ BE THE UNIQUE ELEMENT OF $\mathcal{S}pl_M$ SUCH THAT $s(\rho^\frown\langle i\rangle) \supseteq s(\rho)^\frown\langle i\rangle$ AND $dom(s(\rho^\frown\langle i\rangle))$ IS MINIMAL POSSIBLE AMONG ALL $dom(t)$, $t \in \mathcal{S}pl_M$, $t \supseteq s(\rho)^\frown\langle i\rangle$.

FOR EACH $f \in {}^{\omega}2$, LET

$$s(f) = \bigcup_{n \in \omega} s(f \restriction n).$$

IT IS EASY TO CHECK THAT ALL $s(\rho)$ FOR $\rho \in {}^{<\omega}2$ BELONG TO T: SINCE EACH $s(\rho) \in F_M$, $s(\rho) \in cl(T)$ AND FROM THE ASSUMPTION "NOT CASE 1", $s(\rho) \in T$.

HOWEVER, IT IS ALSO EASY TO CHECK THAT FOR EACH $f \in {}^{\omega}2$, $s(f) \in F_M$. SINCE $|T| < 2^{\omega}$, THERE MUST BE SOME $f \in {}^{\omega}2$ WITH $s(f) \in cl(T) \setminus T$. BUT THIS CONTRADICTS THE ASSUMPTION "NOT CASE 1".

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THEOREM (SHELAH). ASSUME $\mathfrak{s} = \mathfrak{a}$ AND ASSUME THAT THERE IS A FAMILY $\mathcal{P} \subseteq [\mathfrak{s}]^{\omega}$ SUCH THAT $|\mathcal{P}| = \mathfrak{s}$ AND FOR EACH $Z \in [\mathfrak{s}]^{\mathfrak{a}}$ THERE IS SOME $P \in \mathcal{P}$ WITH $|P \cap Z| = \omega$. THEN $\mathrm{RPC}(\omega)$ HOLDS.

THEOREM (SHELAH). ASSUME $\mathfrak{s} > \mathfrak{a}$ AND ASSUME THAT THERE IS A FAMILY $\mathcal{P} \subseteq [\mathfrak{s}]^{\omega}$ SUCH THAT $|\mathcal{P}| = \mathfrak{s}$ AND FOR EACH $Z \in [\mathfrak{s}]^{\mathfrak{a}}$ THERE IS SOME $P \in \mathcal{P}$ WITH $|P \cap Z| = \omega$, HAVING MOREOVER SOME SPECIAL STRUCTURE.
THEN $\mathrm{RPC}(\omega)$ HOLDS.

COROLLARY. $2^{\omega} < \aleph_{\omega} \rightarrow \mathrm{RPC}(\omega)$.