THEOREM (SHELAH). ASSUME $\kappa < \alpha$. THEN RPC($\omega$) HOLDS.

PROOF. LET $\mathcal{D}$ BE AN ARBITRARY DENSE SUBSET OF $[\omega]^\omega$. WE WILL FIND A COMPLETELY SEPARABLE MAD FAMILY $\mathcal{A} \subseteq \mathcal{D}$.

FIX A SPLITTING FAMILY $\{C_\alpha : \alpha < \delta\}$ AND DENOTE $C_\alpha^1 = C_\alpha, C_\alpha^0 = \omega \setminus C_\alpha$.

CONSIDER A TREE $T \subseteq 2^\omega$.

NOTATION:

$succ(T) = \{\sigma \in T : \text{dom}(\sigma) \text{ is a successor ordinal}\}$,

$cl(T) = \{\sigma \in 2^\omega : \forall \alpha \in \text{dom}(\sigma) \exists \delta \in T\}$.

THE PROOF WILL GO BY INDUCTION TO $2^\omega$. WE SHALL BUILD A
A sequence of pairs 
\[ \langle T_\eta, A_\eta : \eta \in \text{succ}(T_\eta) \rangle \] for \( \eta < 2^\omega \) in such a way that 
\[ |T_\eta| \leq \omega + 1 \] and for each 
\( \xi < \eta < 2^\omega \), \( T_\eta \) is a subtree of \( T_\xi \).

We demand:

a) If \( \xi, \eta \in \text{succ}(T_\eta) \) are such that \( \xi \leq \eta \), then \( |A_\xi \cap A_\eta| < \omega \);

b) For each \( \xi \in \text{succ}(T_\eta) \) and for each \( \alpha \in \text{dom}(\xi) \), \( A_\xi \subseteq \mathcal{C}_\alpha^{1,2}(\xi) \).

Given \( \xi \in \text{succ}(T_\eta) \), a set of candidates will be
\[ \mathcal{K}_\eta = \{ \mathcal{D} \in \mathcal{G} : \forall \alpha \in \text{dom}(\xi) \}
\]
Enumerate $[\omega]^{\omega} = \langle M_\eta : \eta < 2^{\omega} \rangle$ in such a way that each set appears $2^{\omega}$-times.

Start with $T_0 = \emptyset$.

If $\eta < 2^\omega$ is a limit ordinal, let $T_\eta = \bigcup \{ T_\xi : \xi < \eta \}$.

Successor step: we know $\langle T_\eta, \langle A_\alpha : \alpha \in \text{succ}(T_\eta) \rangle \rangle$ and $M_\eta$.

If there is a finite set $\{ A_1, A_2, \ldots, A_n \} \subseteq \text{succ}(T_\eta)$ such that $M_\eta \subseteq A_1 \cup A_2 \cup \ldots \cup A_n$, then let $T_{\eta+1} = T_\eta$.

In the opposite case we need to extend $T_\eta$ by adding some $\alpha$ and $A_\alpha$ with $A_\alpha \subseteq M_\eta$. 
FORGET THE SUBSCRIPT: $T = T_2$, $M = M_2$

DENOTE BY $J$ THE IDEAL GENERATED BY $\{A_\lambda : \lambda \in \text{succ}(T)\}$.

LET US CONSIDER TWO SUBSETS OF $T$, DEPENDING ON $M$:

$F_M = F_M^1 \cup F_M^2$, WHERE

$F_M^1 = \{\lambda \in \text{cl}(T) : (\forall X \in J) [M \setminus X]^\omega \cap X_\lambda \neq \emptyset\}$,

$F_M^2 = \{\lambda \in \text{cl}(T) : \text{THE SET } t \in T : A_\lambda \neq \emptyset \& \lambda \in t \& |A_\lambda \cap M| = \omega \text{ IS INFINITE}\}$,

$\phi_M = \{\lambda \in T : \lambda^{<0} \in F_M \& \lambda^{<1} \in F_M\}$.

THERE ARE SEVERAL CASES TO CONSIDER:
CASE ①: THERE IS SOME $α \in F_1^M$ BELONGING TO $\text{cl}(T) \setminus T$;

CASE ②: NOT CASE ① AND $F_1^M = \emptyset$;

CASE ③: NONE OF THE ABOVE.

IN CASE ①, THE $α$ MUST BE IN $F_1^M$ ACCORDING TO THE DEFINITIONS OF $\text{cl}(T)$ AND OF $F_1^M$.

SUBCASE ①A: $\text{dom}(s)$ IS A SUCCESSOR ORDINAL.

Pick some $M' \in [M]^{\omega}$, THEN PICK $A_0 \in [M']^{\omega}$ SUCH THAT $M' \cap A_0$ IS INFINITE. THEN EXTEND $T$ TO $T \cup \{α\}$. 
Subcase 1B: $\text{dom}(A)$ is a limit ordinal.

Let $\alpha = \text{dom}(A)$. Since there is an infinite set in $J_0^\alpha$, $\alpha < \omega$.

For each $x \in J$, we know that $[(M \setminus X)^0_x] \cap J_0^\alpha \neq \emptyset$. Since $C_0^0 \cup C_0^1 = \omega$, there must be some $i \in \{0, 1\}$ with $[(M \cap C_0^i) \setminus X]^0_x \cap J_0^\alpha \neq \emptyset$ whenever $x \in J$. As before, let $t = A^0_x \langle i \rangle$, $A_t \in [(M \cap C_0^i)]^0_x$. Be such that $A_t \in J_0^\alpha$ and there is some $M' \in J_0^\alpha$, $M' \setminus A_t$ infinite, $A_t \preceq M' \subseteq M$. Extend $t$ to $T \cup \{A_t, t\}$. 
SO, IN BOTH SUBCASES WE SUCCEEDED TO FIND SOME $A_\lambda \subseteq M$.

**CASE 2.** \textcolor{red}{SINCE $\nexists \alpha \in \text{Spec}_M$, LET}

\[ S = \bigcup \{ \alpha : \alpha \in F_M \} . \]

**SUBCASE 2A:** \textcolor{red}{$S \in F^1_M$.}

SINCE WE ASSUME NOT CASE 1, WE HAVE $F_M \subseteq T$ AND SO $\text{dom}(S) < \delta$.

PUT $\alpha = \text{dom}(S)$. DECOMPOSE

\[ M = M_0 \cup M_1 \cup M_2, \text{ WHERE} \]

$M_2 = M \cap A_\alpha$, IF $A_\alpha$ IS DEFINED, I.E;

$M_2 = \emptyset$ OTHERWISE;

$M_0 = (M \setminus M_2) \cap C_\alpha$;

$M_1 = (M \setminus M_2) \cap C_\alpha$.

SINCE $S \in F^1_M$, WE HAVE THAT \((\forall X \in J) [M \setminus X]^{\omega} \neq \emptyset \).
Hence the same must hold for one set $M_i$. But $M_2 \subseteq A_2^\omega$ and so $[M_2 \setminus A_2^\omega] = \emptyset$. Since $s \not\in \mathcal{M}$, it cannot hold simultaneously for both 0, 1.

So there is precisely one $i \in \{0, 1\}$ with the property that $(\forall x \in j) [M_i \setminus x] \cap J_x \neq \emptyset$.

As $M_i \cap A_8 = \emptyset$, we have that $s^x \langle i \rangle \in F_{M_i}$. Since $M_i \subseteq M$, $F_{M_i} \subseteq F_M$. So $s^x \langle i \rangle \in F_M$, which contradicts the definition of $s$.

So the subcase 2A never happens.
Subcase 2B: $\check{z} \notin F_m^1$.
$\alpha = \text{dom}(\check{z})$ must be a limit ordinal in this case; $\alpha \leq \delta$.
For each $\alpha < \check{z}$ consider the node $\check{z} |_{\check{\alpha} \wedge 1} < \check{z}(\alpha) = t_\alpha$. Since $\emptyset \in F_m$, $t_\alpha \in F_m$. In particular, $t_\alpha \in F_m^1$. So there is a finite family $\mathcal{A}_\alpha \subseteq \{A_\alpha : A \in \text{succ}(T)\}$ such that $[M \setminus \bigcup \mathcal{A}_\alpha] \cap X_\alpha = \emptyset$.
Then the family $\bigcup_{\alpha < \check{z}} \mathcal{A}_\alpha \cup \{A_\alpha : \alpha \in \check{z}\}$ is of size $\leq |\check{z}| \leq \delta < \delta$, so its trace on $M$ cannot be a mad family on $M$. So there is an infinite $M' \subseteq M$ such that $|M' \cap A| < \omega$ for all $A$ belonging
to $\bigcup \{ A_\alpha : \alpha < 2 \}$. 

CLEARLY, AS $M' \leq M$, WE HAVE ALSO $F^1_{M'} \subseteq F^1_M$ AND $F^2_{M'} \subseteq F^2_M$.

NOW, WE ALWAYS REACH A CONTRACTION:

- IF THERE IS SOME $\alpha < \bar{\alpha}$ SUCH THAT $C_\alpha$ Splits $M'$. FOR A MINIMAL $\alpha$ WITH THIS PROPERTY WE HAVE THAT FOR EACH $x \in J$, $M' \setminus x$ IS FINITE AND CONSEQUENTLY $\exists \alpha^x < 0 \in F^1_{M'}$ AND $\exists \alpha^x < 1 \in F^1_{M'}$, TOO. BUT THIS CONTRADICTS TO THE ASSUMPTION $\text{Sp}_M = \emptyset$.

- IF NO $\alpha < \bar{\alpha}$ WITH $C_\alpha$ SPLITTING $M'$ EXISTS, THEN:
- If $\alpha = \delta$, we have a contradiction that \( \{ C_\alpha : \alpha < \delta \} \) is a splitting family.

- If $\alpha < \delta$, then either

\[
[M' \cap C_\alpha^0] \cap K_{\delta^+}^0 \neq \emptyset, \text{ or }
\]
\[
[M' \cap C_\alpha^1] \cap K_{\delta^+}^1 \neq \emptyset.
\]

In both cases we have a contradiction with the definition of $\delta$.

So case $\delta$ never happens.

**Case 3.** Notice that "not case 0" for $M'$ implies "not case 0" for each infinite $M' \subseteq M$. We have proved already...
That case 2 never happens. So having not case 2 we know that for each infinite 
\( M' \subseteq M \), \( \text{Spec}_M \neq \emptyset \).

Let us define for each \( n \in \omega \) and for each \( g \in \pi^2 \) a mapping \( \Delta(g) \in \text{Spec}_M \) by induction as follows:

\( \Delta(\emptyset) \) is the unique element of \( \text{Spec}_M \) with the minimal \( \text{dom} \).

Suppose \( \Delta(g) \) is known, \( i \in \{0, 1\} \). Let \( \Delta(g^{\langle i \rangle}) \) be the unique element of \( \text{Spec}_M \) such that \( \Delta(g^{\langle i \rangle}) \geq \Delta(g)^{\langle i \rangle} \) and \( \text{dom}(\Delta(g^{\langle i \rangle})) \) is minimal possible among all \( \text{dom}(t), t \in \text{Spec}_M, t \geq \Delta(g)^{\langle i \rangle} \).
For each \( f \in 2 \), let
\[ s(f) = \bigcup_{m \in \mathbb{N}} s(f \upharpoonright m). \]

It is easy to check that all \( s(p) \) for \( p \in \omega \) belong to \( I \): since each \( s(p) \in \mathcal{E}_M \), \( s(p) \in cl(T) \) and from the assumption "not case 1", \( s(p) \in I \).

However, it is also easy to check that for each \( f \in 2 \), \( s(f) \in \mathcal{E}_M \). Since \( |I| < 2^\omega \), there must be some \( f \in 2 \) with \( s(f) \in cl(T) \setminus I \). But this contradicts the assumption "not case 1".
Theorem (Shelah). Assume \( \delta = \alpha \) and assume that there is a family \( \mathcal{F} \subseteq [\delta]^\omega \) such that \( |\mathcal{F}| = \delta \) and for each \( \beta \in [\alpha]^\omega \) there is some \( \mathcal{P} \in \mathcal{F} \) with \( |\mathcal{P} \cap \beta| = \omega \), then \( \text{RPC}(\omega) \) holds.

Theorem (Shelah). Assume \( \delta > \alpha \) and assume that there is a family \( \mathcal{F} \subseteq [\delta]^\omega \) such that \( |\mathcal{F}| = \delta \) and for each \( \beta \in [\alpha]^\omega \) there is some \( \mathcal{P} \in \mathcal{F} \) with \( |\mathcal{P} \cap \beta| = \omega \), having moreover some special structure. Then \( \text{RPC}(\omega) \) holds.

Corollary. \( 2^\omega \not\subseteq \chi_\omega \to \text{RPC}(\omega) \).