COROLLARY. EVERY POINT OF $\omega^*$ IS A $\omega_1$-POINT.

DEFINITION. LET $\mathcal{A}$ BE AN INFINITE ALMOST DISJOINT FAMILY CONSISTING OF COUNTABLE SUBSETS OF SOME SET $X$.

$$J^+(\mathcal{A}) = \{ M \subseteq X : \{ A \in \mathcal{A} : |A \cap M| = \omega \} \geq \omega \}$$

WHAT ARE THE LARGEST FAMILIES, WHICH HAVE AN ALMOST DISJOINT REFINEMENT? WHAT IS THE STRONGEST STATEMENT ABOUT THE EXISTENCE OF AN ALMOST DISJOINT REFINEMENT?

OBSERVATION 13. SUPPOSE $M \subseteq [X]^{\omega_1}$ HAS AN ALMOST DISJOINT REFINEMENT. THEN THERE IS AN ALMOST DISJOINT FAMILY $\mathcal{B}$ WITH $M \subseteq J^+(\mathcal{B})$. 
PROOF. SUPPOSE THAT $A$ IS AN ALMOST DISJOINT REFINEMENT OF $M$. FOR EACH $A \in A$, CHOOSE AN INFINITE ALMOST DISJOINT FAMILY $B(A)$ CONSISTING OF INFINITE SUBSETS OF $A$. PUT $B = \bigcup \{B(A) : A \in A\}$. WHENEVER $M \supseteq A$, THEN $M \cap B$ IS INFINITE FOR EACH $B \in B(A)$. SO $M \subseteq J^+(B)$. 

FOR A CARDINAL $\kappa$, ABBREVIATE: 

$\text{RPC}(\kappa) \equiv \text{ "FOR EVERY INFINITE ALMOST DISJOINT FAMILY } A \subseteq [\kappa]^\omega, J^+(\mathcal{A}) \text{ HAS AN ALMOST DISJOINT REFINEMENT."} $

THIS IS THE STATEMENT WE LOOKED FOR.
DEFINITION [HECHLER 1971] AN ALMOST DISJOINT FAMILY $\mathcal{A} \subseteq [\kappa]^\omega$ IS CALLED COMPLETELY SEPARABLE, IF $\mathcal{A}$ IS INFINITE AND FOR EACH $\mathbf{M} \in \text{J}^+(\mathcal{A})$, THERE IS SOME $\mathbf{A} \in \mathcal{A}$ WITH $\mathbf{A} \setminus \mathbf{M}$, I.E., $\mathcal{A}$ IS AN ALMOST DISJOINT REFINEMENT OF $\text{J}^+(\mathcal{A})$.

OBSERVATION 14. SUPPOSE $\mathcal{A}$ IS A COMPLETELY SEPARABLE ALMOST DISJOINT FAMILY. THEN $|\mathcal{A}| \geq 2^\omega$.

PROOF. CHOOSE $\{\mathbf{A}_n : n \in \omega\} \subseteq \mathcal{A}$. THIS IS POSSIBLE, SINCE $\mathcal{A}$ IS INFINITE. FOR EACH $n \geq 1$, SPLIT $\mathbf{A}_n$ INTO $2^n$ DISJOINT INFINITE PARTS AND ENUMERATE THEM AS $\Sigma_{M_{\mathbf{A}} : \mathbf{A} \in 2^n}$. FOR EACH $\mathbf{f} \in 2^n$,
Let $M_f = \bigcup \{ M_{f \upharpoonright n} : 1 \leq n < \omega \}$. Each set $M_f$ belongs to $J^+(\mathcal{A})$, so there is some $A_f \in \mathcal{A}$ with $A_f \subseteq M_f$. The family $\{ A_f : f \in 2^\omega \}$ is of size $2^\omega$.

Observation 15. Let $\mathcal{A}$ be a completely separable almost disjoint family.

(i) If $M \in J^+(\mathcal{A})$, then both families
\[
\{ M \cap A : A \in \mathcal{A} \text{ and } |A \cap M| = \omega \},
\{ A \in \mathcal{A} : A \subseteq M \}
\]
are completely separable.

(ii) If $\mathcal{A}' \subseteq \mathcal{A}$ satisfies $|\mathcal{A}'| < 2^\omega$, then $\mathcal{A} \setminus \mathcal{A}'$ is completely separable.
(iii) If for each $A \in \mathcal{A}$, $B(A) \subseteq [A]^{\omega}$, the family $\{B(A) : A \in \mathcal{A}\}$ is completely separable.

Proof. Trivial. □

Observation 16. Let $\mathcal{A}$ be a completely separable almost disjoint family. Then for every decreasing sequence $X_0 \supseteq X_1 \supseteq X_2 \supseteq \ldots \supseteq X_n \supseteq \ldots$ of sets from $\mathcal{J}^+(\mathcal{A})$, there is some $Y \in \mathcal{J}^+(\mathcal{A})$ satisfying $Y \subseteq X_n$ for each $n \in \omega$.

Proof. Proceed by induction. By complete separability, choose for each $n \in \omega$ a set $A_n \in \mathcal{A}$ with $A_n \subseteq X_n \setminus \bigcup_{i<n} A_i$. Put $Y = \bigcup_{n \in \omega} A_n$. □
Observation 17. Let $B$ be an infinite almost disjoint family such for each $x \in J^+(B)$, $|\{B \in B : |B \cap x| = \omega\}| = 2^\omega$. Then there is a completely separable almost disjoint family $A$ satisfying $J^+(A) = J^+(B)$.

Proof. For each $x \in J^+(B)$ let $B(x) \in B$ be such that $B(x) \cap x$ is infinite, $B(x) \neq B(x')$ for distinct $x, x' \in J^+(B)$.

Let $A$ consist of all $x \in B(x)$ for $x \in J^+(B)$. \( \square \)

Definition. A chain of length $\gamma$ is a family $\mathcal{F} = \{T_\alpha : \alpha < \gamma\} \subseteq [\omega]^\omega$ satisfying $T_\alpha \subseteq T_\beta$ whenever $\alpha < \beta < \gamma$. 
Two chains $T$ and $T'$ are called disjoint if there are $T \subseteq T$ and $T' \subseteq T'$ with $T \cap T'$ finite.

Given a chain $T$ and a set $X$, let us say

$X$ is below $T$ if $x \in T$ for each $T \subseteq T$.

$X$ is compatible with $T$ if $X \cap T$ is infinite for each $T \subseteq T$.

$X$ meets the boundary of $T$ if for each $T \subseteq T$ there is some $T' \subseteq T$ with $X \cap (T \setminus T')$ infinite.

Observation 18. Let $y$ be an ordinal of countable cofinality, let $T = \{T_\alpha : \alpha < \delta \}$ be a chain. Then there is a family of chains
\{ \mathcal{T}_k : \mathcal{F} < \mathcal{B}, \mathcal{F}(\mathcal{F}) = \omega \} \text{ with each }
\mathcal{T}_k \text{ of length } \omega + \omega, \ \mathcal{T} \subseteq \mathcal{T}_k \text{, such that for each } x \in [\omega]^\omega \text{, if } x \text{ meets the boundary of } \mathcal{T}, \text{ then } |\{ \mathcal{F} < \mathcal{B} : x \text{ meets the boundary of } \mathcal{T}_k \}| = \mathcal{B}.

\text{Proof: Choose a sequence } 
\mathcal{F}_0 < \mathcal{F}_1 < \mathcal{F}_2 < \ldots < \mathcal{F}_n < \ldots \text{ of type } \omega, \text{ cofinal in } \mathcal{F}, \text{ for } n \in \omega, \text{ let } 
\mathcal{R}_n = \bigcap_{i < n} \mathcal{T}_{\mathcal{F}_i} \setminus \mathcal{T}_{\mathcal{F}_n}. \text{ The family } 
\{ \mathcal{R}_n : n < \omega \} \text{ is pairwise disjoint and consists of infinite sets.} 
\text{Let } \mathcal{R}_n = \{ \mathcal{R}(n, k) : k \in \omega \}. 
\text{For a mapping } \mathcal{F} : \omega \to \omega, \text{ put } 
\mathcal{B}(\mathcal{F}) = \{ \mathcal{R}(n, k) : n \in \omega, k < \mathcal{F}(n) \}.\}
Choose a family $\{f_\xi : \xi < \omega\}$ with no upper bound in the order $\varepsilon^*$, and such that for each $\xi < \eta < \xi$, $B(f_\eta) \setminus B(f_\xi)$ is infinite, and each $f_\xi$ is increasing. For each $\xi < \omega$ of countable cofinality, select a strictly increasing sequence $\langle f_i : i \in \omega \rangle$ converging to $f_\xi$ and define

$$T_\xi = T \cup \{B(f_\xi) \setminus \bigcup_{i < j} B(f_{\xi_j}) : j \in \omega\}$$

Clearly, each $T_\xi$ is a chain of length $\omega + \omega$.

Suppose $X \subseteq \omega$ meets the boundary of $T$. Hence $X$ has an infinite intersection with infinitely many $R_n$'s. The same is true for the
set $X \setminus B(f)$, with an arbitrary $f \in \omega_0$, because for each $f$ and for each $n \in \omega$, $R_n \cap B(f)$ is finite.

Let $\eta < \theta$ be arbitrary. Find $\xi_0 > \eta$ such that $(X \setminus B(f_\xi)) \cap B(f_{\xi_0})$ is infinite. Induction: knowing $\xi_i$, choose $\xi_{i+1}$ such that the set $(X \setminus B(f_{\xi_i})) \cap B(f_{\xi_{i+1}})$ is infinite. Since $\theta$ is regular uncountable, $\sup_{i \in \omega} \xi_i = \xi < \theta$ and the construction guarantees that $X$ meets the boundary of $f_\xi$. \qed
THEOREM. THERE EXISTS A COMPLETELY SEPARABLE ALMOST DISJOINT FAMILY.

PROOF. CONSIDER A TREE OF HEIGHT $\omega$, CONSISTING OF ALL MAPPINGS $\delta: \alpha \rightarrow \beta$ WITH ALL VALUES $\delta(\beta)$ LIMIT ORDNALS OF COUNTABLE COFINALITY; CALL THIS TREE $\Gamma$.

FOR EACH $\delta \in \Gamma$, WE SHALL FIND A TOWER $T_\delta$ OF COUNTABLE LENGTH AND A SET $A_\delta$, WHICH IS BELOW $T_\delta$. THE FAMILY $\{A_\delta: \delta \in \Gamma\}$ WILL BE AS REQUIRED.

PROCEED BY TRANSFINITE INDUCTION TO $\omega$.

START: LET $T_\varnothing$ BE AN ARBITRARY TOWER OF LENGTH $\omega$, LET $A_\varnothing$ BE AN ARBITRARY SET BELOW $T_\varnothing$. 
INDUCTION STEP, $\alpha < \omega_1$, $\alpha = \beta + 1$:
we know all towers $\gamma_{\alpha}$ for $\gamma : \beta \rightarrow \gamma$, $\alpha \in \Gamma$, and all sets $A_\alpha$, each $A_\alpha$ below $\gamma_{\alpha}$.

Given $\alpha$, we have to define towers $\gamma_{\alpha} \cap \xi$ and sets $A_\alpha \cap \xi$ for all $\xi < \beta$, $\gamma(\xi) = \omega$.

Since $A_\alpha$ is below $\gamma_{\alpha}$, we shall apply Observation 18, choosing the unbounded family $\{ \xi : \xi < \beta \}$ in such a way that $A_\alpha \subseteq B(\xi_0)$.

Knowing all $\gamma_{\alpha}$ for $\gamma : \alpha \rightarrow \beta$, $\alpha \in \Gamma$, choose $A_\beta$ infinite and below $\gamma_{\alpha}$ arbitrarily.
**Induction Step, $\alpha < \omega_1$, $\alpha$ Limit:**

Knowing all $T_\beta$ and $A_\beta$ for all $\beta \in \Gamma$ with $\text{dom}(\beta) \subset \alpha$,

define for $\alpha : \alpha \rightarrow \beta$, $\alpha \in \Gamma$, the tower $T_\alpha$ simply as $\bigcup_{\beta < \alpha} T_\beta$.

It remains to find sets $A_\alpha$.

Denote by $\mathcal{X}$ the family of all $x \in \omega$ such that the family

$\{ T_\alpha : \alpha \in \Gamma, \text{dom}(\alpha) = \alpha, \text{the set } x \text{ meets the boundary of } T_\alpha \}$

is of size $2^\omega$.

Assign to each $x \in \mathcal{X}$ one $\alpha : \alpha \rightarrow \beta$, $\alpha \in \Gamma$, such that $x$ meets the boundary of $T_\alpha$ in a one-to-one way, denote this $\delta$ as $\lambda(x)$. 
Then let $A_\alpha$ be an arbitrary infinite set below $\cap_\alpha$ for all $\alpha \subseteq \{\lambda(x) : x \in \mathcal{X}\}$, for $x \in \mathcal{X}$ we demand moreover $A_\alpha \subseteq x$.

This works.

The family $\{A_\alpha : \alpha \in \Gamma\}$ is almost disjoint:

Suppose $\alpha, \beta \in \Gamma$, act:

Denote $\alpha = \text{dom}(\alpha)$, by induction step $\alpha \mapsto \alpha + 1$, we choose for extending $\cup_\alpha$ a family $\{f_\beta : \beta \in \Gamma\}$ with $A_\alpha \subseteq B(f_\alpha)$. The construction in observation 18 guarantees that each tower $\cap_\alpha$ contains some member disjoint with $B(f_\alpha)$ so $A_\alpha \subseteq B(f_\alpha)$, $A_\alpha \cap B(f_\alpha)$ is finite.
Suppose $\alpha, \beta \in \Gamma$ are incomparable. Choose minimal $\alpha < \omega_1$ satisfying $\delta^\alpha \neq \varepsilon^\alpha$. This $\alpha$ must be a successor ordinal, $\alpha = \beta + 1$.

Denote by $\rho = \delta^\beta = \varepsilon^\beta$. Then $\Gamma_\rho \supseteq \Gamma_{\delta^\beta}$ and $\Gamma_\rho \supseteq \Gamma_{\varepsilon^\beta}$.

Assume $\delta(\beta) < \varepsilon(\beta)$. Then the set $A_\rho \subseteq \mathcal{B}(f_{\delta(\beta)})$ and $A_\beta \subseteq \mathcal{B}(f_{\varepsilon(\beta)}) \setminus \mathcal{B}(f_{\delta(\beta)})$.

The family $\{A_\rho : \rho \in \Gamma\}$ is completely separable: suppose $X \subseteq \omega_1$, $\{\rho \in \Gamma : |X \cap A_\rho| = \omega\}$ is infinite. Then there is some $\xi \in \Gamma$ such that $X$ meets the boundary of $\Gamma_\xi$. 
To see this, let $\alpha < \omega$, be the smallest one with the set $K = \{ \delta \in \Gamma : |X \cap A_\delta| = \omega \text{ and } \text{dom}(\delta) \subseteq \alpha \}$ infinite.

If $\alpha = \beta + 1$, then for some $\xi \in \Gamma$, $\text{dom}(\xi) = \beta$, $\{ \delta \in K : \xi < \delta \}$ is infinite. By the construction, there is an increasing sequence of limit ordinals in $\Gamma$, $\langle \xi_n : \text{new} \rangle$ with each $\xi_n \in K$. The unbounded family of mappings in $\omega^\omega$ was $\{ \xi : \xi < \xi \}$; the set $B(\xi_n) \setminus B(\xi_{n-1}) \in \tilde{T}_\xi \tilde{T}_{\xi_n}$ and $A_{\tilde{T}_\xi \tilde{T}_{\xi_n}}$ is below $\tilde{T}_\xi \tilde{T}_{\xi_n}$. Therefore for $\xi' = \sup \{ \xi_n : \text{new} \}$ we have that $X$ meets the boundary of $\tilde{T}_\xi \tilde{T}_{\xi'}$. 
Since for each $n$, $|X \cap A_{\gamma_n}| = \omega$.

If $\alpha$ is a limit ordinal, consider the set $\{ \xi \in \Gamma : \text{for some } \delta \in K, A_\delta \text{ is below } \xi \} = T$.

Since $T$ is countable tree and all levels of $T$ except possibly the last one are finite, there is a cofinal branch in $T$.

Choosing $t_0, t_1, c_{t_1}, ..., c_{t_n}, c_{t_n} \ldots \text{ in } T$, cofinal part of this branch, then the tower $\{ t_\delta \in \Gamma \}$ for $\delta = \bigcup t_n$ satisfies that $X$ meets the boundary of $T$.

We have verified that for $X \in J^+(\{A_\delta : \delta \in \Gamma\})$ there is some $\delta \in \Gamma$ such that $X$ meets
MEETS THE BOUNDARY OF $\mathcal{T}_\alpha$.

Suppose $\text{dom}(\Delta) = \alpha$. By Observation 18, there are $\mathfrak{b}$-many $t \leq \Delta$ with $\text{dom}(t) = \alpha + 1$ such that $X$ meets the boundary of $\mathcal{T}_t$, and for each such $t$ there are $\mathfrak{b}$-many $p \geq t$ with $\text{dom}(p) = \alpha + 2$ such...

So when constructing all $A_\Delta$ with $\text{dom}(\Delta) = \alpha + \omega$, we have $X \in \mathcal{X}$ in this step of recursion. So for some $\Delta$ with $\text{dom}(\Delta) = \alpha + \omega$ we have guaranteed that $A_\Delta \subseteq X$. \qed
Corollary. Let $E$ be a countable almost disjoint family, let $\mathcal{B}$ be a dense subset of $\mathcal{P}(\omega)/\text{fin}$. Then there is a completely separable almost disjoint family $A \subseteq E$, such that for each $A \in E$, $A^* \in \mathcal{D}$. Moreover, $A$ refines $\mathcal{J}(E)$.

Theorem. Assume $\omega = \omega_1$. Then $\text{RPC}(\omega)$ holds true.

Proof. Let $\mathcal{B}$ be an arbitrary mad family on $\omega$. Fix a splitting family $\{Q_\alpha : \alpha < \omega_1\}$; denote $Q_\alpha(0) = Q_\alpha$, $Q_\alpha(1) = \omega \setminus Q_\alpha$. For each $\alpha < \omega_1$ and each mapping $\delta : \alpha \to 2$, the filter $\mathcal{F}_\delta$, generated by $\{Q_\beta(\delta(\beta)) : \beta < \alpha\}$ has a
COUNTABLE BASIS, so there is a TOWER \( \tau \), which is a basis of \( \tau_a \).

For each \( \alpha < \omega_1 \) and for each \( s: \alpha \rightarrow 2 \), there is a completely separable almost disjoint family \( D_\alpha \), consisting of sets below \( \tau_a \) and refining all sets, which meet the boundary of \( \tau_a \). Put

\[ D_\alpha = \bigcup \{ D_\beta : \beta \in a \times 2 \} \]

THE FAMILY \( D_\alpha \) IS COMPLETELY SEPARABLE. WE CAN ALSO CHOOSE \( D_\alpha \) IN SUCH A WAY THAT FOR EACH \( D \in D_\alpha \), THERE IS A UNIQUE \( B \in B \) SATISFYING \( D \subseteq B \).

PUT \( A_\alpha = D_\alpha \) AND FOR \( \alpha < \omega_1 \), LET \( A_\alpha = \{ D \in D_\alpha : (\forall \beta < \alpha)(\forall A \in A_\beta)(D \cap A) \) IS FINITE. \}
The family \( A = \bigcup_{\alpha < \omega_1} A_\alpha \) is the required almost disjoint refinement of \( J^+(B) \). Obviously, \( A \) is almost disjoint. So if \( M \in J^+(B) \) we have to find some \( A \in A \) with \( A \subseteq M \).

However, we need less. It is enough to prove that for some \( \alpha < \omega_1 \), \( M \in J^+(A_\alpha) \).

To see this, suppose that for some \( \alpha \), \( M \in J^+(A_\alpha) \). Choose the \( \alpha \) to be the smallest one.

If \( \alpha = 0 \), then \( A_0 = B_0 \) and \( B_0 \) is completely separable, hence \( M \) contains some \( D \subseteq D_0 = A_0 \).

If \( \alpha > 0 \), then for each \( \beta < \alpha \),
The set \( \{ A \in \mathcal{A}_\beta : |A \cap M| = \omega \} \) is finite.

There is an infinite countable subset \( \mathcal{A}' \subseteq \mathcal{A}_\beta \) such that for each \( A' \in \mathcal{A}' \), \( |A' \cap M| = \omega \) and by the choice of \( \mathcal{A}_\beta \), \( A' \cap A \) is finite for each \( A \in \bigcup_{\beta < \kappa} \mathcal{A}_\beta \).

Consequently, there is a subset \( M' \subseteq M \), such that \( M' \) is almost disjoint with all \( A \in \bigcup_{\beta < \kappa} \mathcal{A}_\beta \) and \( M' \cap A' \) is infinite for each \( A' \in \mathcal{A}' \). Hence \( M' \in J^+(\mathcal{A}_\beta) \), so for some \( D \in \mathcal{A}_\beta \), \( D \subseteq M' \).

Now \( D \) must belong to \( \mathcal{A}_\beta \), since it is almost disjoint with all \( A \in \bigcup_{\beta < \kappa} \mathcal{A}_\beta \).
So fix an $M \in \mathcal{J}^+(B)$ and we have to show that for some $\alpha$, $M \in \mathcal{J}^+(A_\alpha)$. Choose $\alpha_0 < \omega_1$ such that for some $\delta : \alpha_0 \to 2$ we have that $M$ meets the boundary of $\mathcal{J}_0$ and that for each $T \in \mathcal{J}_0$, $M \cap T \in \mathcal{J}^+(B)$.

If $M \in \mathcal{J}^+(A_\beta)$, for some $\beta \leq \alpha_0$, we are done. But if $M \notin \mathcal{J}^+(A_\beta)$ for all $\beta \leq \alpha_0$, then there is a countable subset $B_0 \subseteq B$ such that for each $\beta \leq \alpha_0$ and for each $A \in A_\beta$, if $|A \cap M| = \omega$, then $A \subseteq B$ for some $B \in B_0$. There is a set $M' \subseteq M$ with the following properties:
a) \( M' \in \mathcal{J}(\mathcal{B}) \);

b) \( M'_0 \cap B \) is finite for each \( B \in \mathcal{B} \).

c) \( M'_0 \cap \mathcal{T} \) is infinite for each \( T \in \mathcal{J}_{\alpha_0} \).

Observe: The set \( M'_0 \) does not meet the boundary of \( \mathcal{J}_{\alpha_0} \) — in the opposite case the would exist some \( D \in \mathcal{D}_{\alpha_0}, D \subseteq M'_0 \) and since \( D \subseteq B \) for no \( B \in \mathcal{B}_0 \), \( D \subseteq \mathcal{A}_{\alpha_0} \), contrary to our assumption.

So there is some \( T \in \mathcal{J}_{\alpha_0} \) such that \( M'_0 \cap \mathcal{T} \) is below \( \mathcal{J}_{\alpha_0} \).

Choose \( B \in \mathcal{B} \) such that \( M'_0 \cap \mathcal{T} \cap B \) is infinite and put
\[ M_0 = M_0' \cap T \setminus B_0. \]

Let us repeat the previous try with the set \( M_0 \): let \( \alpha_1 > \alpha_0 \)
be such that for some \( \beta_1 > \beta_0 \), \( \beta_1: \alpha_1 \to 2 \) we have that \( M_0 \) meets
the boundary of \( J_{\alpha_1} \) and for each \( T \in J_{\alpha_1} \), \( M_0 \cap T \in J^+(B) \).

Assume that we were unlucky again: for each \( \beta \leq \alpha_1 \), \( M_0 \in J^+(A_{\beta}) \).

Apply the previous reasoning to get countable \( B_1 \subseteq B, B_1 \in B \)
and \( M_1 \subseteq M_0 \), \( M_1 \in J^+(B) \).

Continue: \( \alpha_2 > \alpha_1 \), \( B_2 \subseteq B, B_2 \in B \).

\[ M_2 \subseteq M_1 \subseteq M_0 \]

... etc. If we needed all \( \omega \)-many steps, then we get
\[ \bar{\alpha} = \sup_{\text{new}} \alpha \text{ and } \delta = \bigcup_{\text{new}} \alpha. \]
Since $\Phi(w)/\zeta_n$ has a strong countable separation property, there is a set $L \subseteq M$ such that $L \supseteq M_n \cap B_n$ for all $n \in \omega$ and $|L \cap B| < \omega$ for each $B \in \bigcup_{n \in \omega} B_n$.

Clearly, the set $L \subseteq J^+(B)$ and $L$ meet the boundary of $F_\alpha$, so $L \subseteq J^+(B_\alpha)$. If $D \subseteq B_\alpha$ satisfies $D \subseteq L$, the $D$ is almost disjoint with each $B \in \bigcup_{n \in \omega} B_n$, so $D \subseteq A_\alpha$. Thus $L \subseteq J^+(A_\alpha)$, so $M \subseteq J^+(A_\alpha)$ as well. □
PROBLEM. IS THERE A COMPLETELY SEPARABLE MAD FAMILY?

THEOREM. THE FOLLOWING ARE EQUIVALENT:

(i) $\text{RPC} (\omega)$;

(ii) FOR EACH INFINITE MAD FAMILY $\mathcal{B}$ ON $\omega$, $\mathcal{J}^+ (\mathcal{B})$ HAS AN ALMOST DISJOINT REFINEMENT BY A COMPLETELY SEPARABLE ALMOST DISJOINT FAMILY;

(iii) THERE IS A BASE TREE $T$ SUCH THAT EVERY MAD FAMILY $\mathcal{A}$ IS COMPLETELY SEPARABLE.

PROOF. (iii) $\rightarrow$ (ii) $\rightarrow$ (i) IS TRIVIAL.

(i) $\rightarrow$ (iii): LET $\Theta = \{ \mathcal{A}_\alpha : \alpha < \kappa \}$ BE AN ARBITRARY BASE MATRIX. PROCEEDING BY A TRANSFINITE
INDUCTION, FIND A NEW BASE MATRIX 
\{ B_\alpha : \alpha < h \} SUCH THAT

a) FOR EACH \alpha < h, B_\alpha \subseteq A_\alpha;

b) |B| = 2^\omega;

c) FOR EACH \alpha < h, B_\alpha + 1 IS AN ALMOST DISJOINT REFINEMENT
OF \( J^*(B_\alpha) \) - HERE \( (i) \) IS USED.

FOR EACH \alpha < h AND FOR EACH B \in B_\alpha, CHOOSE A UNIQUE C(B) \in B_\alpha + 1
SATISFYING C(B) \subseteq B.

THE FAMILY \{ C(B) : B \in \bigcup B_\alpha \}_{\alpha < h}
IS DENSE IN \( (P(\omega), \subseteq^*) \) AND
IS A TREE UNDER \( \subseteq^* \).

LET \( D \subseteq \{ C(B) : B \in \bigcup B_\alpha \}_{\alpha < h} \)
BE AN ARBITRARY MAD FAMILY AND
LET $M \in J^+(\emptyset)$ BE ARBITRARY.
FOR EACH $\alpha < \kappa$ LET
$$\mathcal{B}_\alpha(M) = \{B \in \mathcal{B}_\alpha : |B \cap M| = \omega\}.$$
LET $\alpha < \kappa$ BE THE FIRST ONE WITH
$$\bigcup_{\beta < \alpha} \mathcal{B}_\beta(M)$$ INFINITE.

CASE 1: $\alpha = \beta + 1$. WE HAVE $\mathcal{B}_\beta(M)$ INFINITE. THE COLLECTION $\bigcup_{\gamma < \beta} \mathcal{B}_\gamma(M)$ IS FINITE AND $\emptyset$ IS ALMOST DISJOINT, HENCE EACH $B \in \mathcal{B}_\beta(M)$ HAS AN INFINITE INTERSECTION ALSO WITH $M' = M - \bigcup_{\gamma < \beta} \mathcal{B}_\gamma(M)$. SINCE
$$\emptyset \subseteq \{C(B) : B \in \bigcup_{\gamma < \beta} \mathcal{B}_\gamma\},$$
WE GET THAT $M' \in J^+(\mathcal{B}_{\beta-1})$. BY $c)$, THERE IS SOME $B \in \mathcal{B}_\beta$ WITH $B \subseteq M'$. 
Since $B$ is disjoint with $\bigcup_{\delta<\beta} B_\delta(M)$ and since $\emptyset$ is maximal, there must be some $D \in \emptyset$ with $D \subseteq M'$.

Case 2: $\alpha$ is a limit ordinal.

Consider a family $\{B \in B_{\alpha} : \text{for each } \beta<\alpha \text{ and each } B' \in B_\beta(M), B \cap B' \text{ is finite}\} := B_{\alpha}^\prime$. There is a set $M' \subseteq M$ such that $M' \in J^+(B_{\alpha}^\prime)$ and $M' \cap B$ is finite for each $B \in \bigcup_{\beta<\alpha} B_\beta(M)$. By c), there is some $B \in B_{\alpha+1}$ with $B \cap M' \subseteq M'$ and we may continue as in case 1. \qed
Observation 19. Suppose that each dense subset of $([\omega]^\omega, s^*)$ contains a completely separable MAD family. Then $\text{RPC}(\omega)$ holds true.

Indeed, for a MAD family $Q$, consider $\mathcal{D} = \{D \in [\omega]^\omega : \text{for some } Q \in Q, D \in Q\}$. Any completely separable MAD family $A \in \mathcal{D}$ is the almost disjoint refinement of $Q$.

Saharon Shelah proved in 2009 the strongest known statement about $\text{RPC}(\omega)$. 