ALMOST DISJOINT REFINEMENT BY COUNTABLE SETS

THEOREM (BERNSTEIN, SIERPIŃSKI, KURATOWSKI, ...): LET κ BE AN INFINITE CARDINAL, $\{A_\alpha : \alpha < \kappa\}$ A FAMILY OF SETS WITH $|A_\alpha| = \kappa$ FOR EACH α.

THEN THERE IS A PAIRWISE DISJOINT FAMILY $\{D_\alpha : \alpha < \kappa\}$ WITH $|D_\alpha| = \kappa$ AND $D_\alpha \subseteq A_\alpha$ FOR EACH α.

PROOF A STANDARD INDUCTION. □

— <u>A DISJOINT REFINEMENT</u>

DEFINITION. A FAMILY $\mathcal{A}$ IS **AN ALMOST DISJOINT REFINEMENT** OF A FAMILY $\mathcal{M}$, IF FOR EACH $M \in \mathcal{M}$ THERE IS AN $A \in \mathcal{A}$ WITH $A \subseteq M$, EACH $A \in \mathcal{A}$ IS COUNTABLY INFINITE AND ANY TWO DISTINCT MEMBERS OF $\mathcal{A}$ ARE ALMOST DISJOINT.

WHICH SUBFAMILIES OF $[\omega]^{\omega}$ HAVE AN ALMOST DISJOINT REFINEMENT?

THEOREM (BALCAR, VOJTÁŠ 1980) EACH FREE ULTRAFILTER ON ω HAS AN ALMOST DISJOINT REFINEMENT.

THE BASE MATRIX TREE.

A MAD FAMILY ON ω IS A FAMILY $\mathcal{A} \subseteq [\omega]^\omega$ SUCH THAT

(i) ANY TWO MEMBERS ARE ALMOST DISJOINT;

(ii) $\mathcal{A}$ IS A MAXIMAL FAMILY SATISFYING (i).

OBSERVATION 1. THERE IS A COLLECTION $\mathcal{B}$ CONSISTING OF MAD FAMILIES ON ω SUCH THAT FOR EACH $M \in [\omega]^\omega$

(*) THERE IS SOME $\mathcal{A} \in \mathcal{B}$ SUCH THAT M INTERSECTS AT LEAST <u>TWO</u> MEMBERS OF $\mathcal{A}$ IN AN INFINITE SET.

PROOF. FOR $X \in [\omega]^\omega$ CHOOSE AN ARBITRARY MAD $\mathcal{A}(X)$

SUCH THAT $X \in \mathcal{A}(X)$. PUT

$\Theta = \{\mathcal{A}(X) : X \in [\omega]^\omega\}$. GIVEN ARBITRARY $M \in [\omega]^\omega$, SELECT $X \subseteq M$ WITH X AND $M \setminus X$ INFINITE. BY MAXIMALITY OF $\mathcal{A}(X)$, THE SET M MUST MEET ALSO SOME OTHER SET IN $\mathcal{A}(X)$, DISTINCT FROM X. $\square$

DEFINITION. $\mathfrak{h}$ IS THE MINIMAL SIZE OF A FAMILY Θ, CONSISTING OF MAD FAMILIES SATISFYING (*).

OBSERVATION 2. $\omega < \mathfrak{h} \leq 2^\omega$

PROOF. $\mathfrak{h} \leq 2^\omega$ BY OBSERVATION 1. IF $\{\mathcal{A}_n : n \in \omega\}$ IS A COLLECTION OF MAD FAMILIES, PICK $A_n \in \mathcal{A}_n$

SO THAT $\{A_n : n\in\omega\}$ GENERATE A UNIFORM FILTER. THIS IS POSSIBLE BY THE MAXIMALITY OF ALL $\mathcal{A}_n$'S. THEN CHOOSE $x_0 < x_1 < x_2 < \dots < x_n < \dots$ WITH $x_n \in \bigcap_{i\le n} A_i$. THE SET $X = \{x_n : n\in\omega\}$ MEETS ONLY ONE ELEMENT OF $\mathcal{A}_n$ IN AN INFINITE INTERSECTION. SO $\omega < \mathfrak{h}$. □

NOTATION. A FAMILY Θ, CONSISTING OF MAD FAMILIES ON ω

– A MATRIX

A MATRIX, SATISFYING (*)

– A SHATTERING MATRIX

SUPPOSE $\mathcal{A}$, $\mathcal{B}$ ARE TWO MAD FAMILIES ON ω. $\mathcal{A} \prec \mathcal{B}$

($\mathcal{A}$ IS FINER THAN $\mathcal{B}$) IF FOR EACH $A \in \mathcal{A}$ THERE IS A $B \in \mathcal{B}$ WITH $A \subseteq^* B$ ($\equiv A \setminus B$ IS FINITE).

OBSERVATION 3. LET $\kappa < \mathfrak{h}$ AND SUPPOSE THAT $\{\mathcal{A}_\alpha : \alpha < \kappa\}$ IS A MATRIX. THEN THERE IS A MAD FAMILY $\mathcal{B}$ SUCH THAT FOR ALL $\alpha < \kappa$, $\mathcal{B} \prec \mathcal{A}_\alpha$.

PROOF. THERE IS AT LEAST ONE SET $B \in [\omega]^\omega$ SUCH THAT FOR EACH $\alpha < \kappa$, B MEETS ONLY ONE $A \in \mathcal{A}_\alpha$ IN AN INFINITE INTERSECTION. (REASON: $\kappa < \mathfrak{h}$ PLUS MINIMALITY OF $\mathfrak{h}$)

LET $\mathcal{B}$ BE A MAXIMAL FAMILY, WHICH IS ALMOST DISJOINT AND CONSISTS FROM SUCH B'S ONLY.

$\mathcal{B}$ IS A MAD FAMILY: IF NOT,

THEN THERE IS AN INFINITE SET M, SUCH THAT THERE IS <u>NO</u> INFINITE $C \subseteq M$ WITH THE PROPERTY THAT C MEETS ONLY ONE ELEMENT OF $\mathcal{A}_\alpha$ IN AN INFINITE INTERSECTION.

SO $\{A \cap M : A \in \mathcal{A}_\alpha, |A \cap M| = \omega\} = \mathcal{B}_\alpha$ IS A MAD FAMILY ON M AND $\{\mathcal{B}_\alpha : \alpha < \kappa\}$ IS A SHATTERING MATRIX ON M. ANY BIJECTION FROM M ONTO ω SHOWS NOW THAT $\kappa \geq \mathfrak{h}$, A CONTRADICTION.

SO $\mathcal{B}$ IS A MAD FAMILY AND IS FINER THAN ALL $\mathcal{A}_\alpha$'S. $\square$

OBSERVATION 4. THERE IS A SHATTER-ING MATRIX $\{\mathcal{B}_\alpha : \alpha < \mathfrak{h}\}$ WITH THE PROPERTY THAT FOR ANY $\alpha < \beta < \mathfrak{h}$, $\mathcal{B}_\beta \prec \mathcal{B}_\alpha$.

PROOF. TRANSFINITE INDUCTION. CHOOSE ARBITRARILY A SHATTERING MATRIX $\{\mathcal{A}_\alpha : \alpha < \mathfrak{h}\}$ AND PUT $\mathcal{B}_0 = \mathcal{A}_0$. KNOWING $\mathcal{B}_\alpha$ FOR ALL $\alpha < \beta < \mathfrak{h}$, APPLY OBSERVATION 3 TO THE MATRIX $\{\mathcal{A}_\alpha : \alpha \leq \beta\} \cup \{\mathcal{B}_\alpha : \alpha < \beta\}$ TO GET $\mathcal{B}_\beta$. $\square$

NOTATION. A MATRIX $\mathcal{A} = \{\mathcal{A}_\alpha : \alpha < \mathfrak{h}\}$ SUCH THAT FOR EACH $\alpha < \beta < \mathfrak{h}$, $\mathcal{A}_\beta \prec \mathcal{A}_\alpha$ – A **REFINING** MATRIX

OBSERVATION 5. $\mathfrak{h}$ IS A REGULAR CARDINAL.

PROOF. BY OBSERVATION 4, THERE IS A MATRIX $\{\mathcal{A}_\alpha : \alpha < \mathfrak{h}\}$ WHICH IS SHATTERING AND REFINING.

LET I BE COFINAL SUBSET OF $\mathfrak{h}$.

THEN THE MATRIX $\{\mathcal{A}_\alpha : \alpha \in I\}$ IS ALSO SHATTERING, SO $|I| = \mathfrak{h}$. $\square$

OBSERVATION 6. LET $\{\mathcal{A}_\alpha : \alpha < \mathfrak{h}\}$ BE A SHATTERING AND REFINING MATRIX, LET $M \in [\omega]^\omega$. THEN THERE IS SOME $\alpha < \mathfrak{h}$ SUCH THAT $|\{A \in \mathcal{A}_\alpha : A \cap M \text{ IS INFINITE}\}| = 2^\omega$.

PROOF. SINCE THE MATRIX IS SHATTERING, THERE IS SOME $\alpha_0 < \mathfrak{h}$ AND TWO DISTINCT $A_0, A_1 \in \mathcal{A}_{\alpha_0}$ SUCH THAT $|A_0 \cap M| = \omega = |A_1 \cap M|$. THE SET $A_0 \cap M$ IS INFINITE, SO THERE IS SOME $\beta_0 < \mathfrak{h}$ AND TWO DISTINCT $A_{00}, A_{01} \in \mathcal{A}_{\beta_0}$ SUCH THAT

$|A_{00} \cap A_0 \cap M| = \omega = |A_{01} \cap A_0 \cap M|$,

SIMILARLY, THERE IS SOME $\beta_1 < \mathfrak{h}$ AND $A_{10}, A_{11} \in \mathcal{A}_{\beta_1}$ SUCH THAT

$$|A_{10} \cap A_1 \cap M| = \omega = |A_{11} \cap A_1 \cap M|$$

PUT $\alpha_1 = \max\{\beta_0, \beta_1\}$. SINCE THE MATRIX IS REFINING, WE CAN ASSUME THAT $A_{00}, A_{01}, A_{10}, A_{11}$ BELONG TO $\mathcal{A}_{\alpha_1}$. CONTINUING BY INDUCTION, WE FIND AN INCREASING SEQUENCE $\langle \alpha_n : n < \omega \rangle$ AND DISTINCT SETS $A_\varphi \in \mathcal{A}_{\alpha_n}$ FOR ALL $\varphi \in {}^{n+1}2$, SUCH THAT $|A_\varphi \cap M| = \omega$ AND FOR $\varphi \subsetneq \psi$, $A_\varphi \supseteq^* A_\psi$. — LET $\alpha = \sup\{\alpha_n : n\}$. IN $\mathcal{A}_\alpha$, FOR EACH $f \in {}^\omega 2$, THERE

IS SOME $A_f \in \mathcal{A}_\alpha$ SUCH THAT

$A_f \subseteq^* A_{f \restriction n}$ FOR ALL $n \in \omega$ AND

$A_f \cap M$ IS INFINITE. $\square$

OBSERVATION 7. THERE IS A SHATTERING AND REFINING MATRIX $\{\mathcal{A}_\alpha : \alpha < \mathfrak{h}\}$ SUCH THAT FOR EACH $M \in [\omega]^\omega$ THERE IS SOME $\alpha < \mathfrak{h}$ AND $A \in \mathcal{A}_\alpha$ WITH $A \subseteq M$.

PROOF. CHOOSE AN ARBITRARY SHATTERING AND REFINING MATRIX $\{\mathcal{B}_\alpha : \alpha < \mathfrak{h}\}$.

TRASFINITE INDUCTION: SUPPOSE $\alpha < \mathfrak{h}$ AND SUPPOSE THAT ALL $\mathcal{A}_\beta$'S FOR $\beta < \alpha$ ARE ALREADY KNOWN.

BY OBSERVATION 3, THERE IS A MAD FAMILY $\mathcal{C}_\alpha$ SUCH THAT

$\mathcal{C}_\alpha \prec \mathcal{B}_\beta$ FOR ALL $\beta \le \alpha$, AND

$\mathcal{C}_\alpha \prec \mathcal{A}_\beta$ FOR ALL $\beta < \alpha$.

CONSIDER $\mathfrak{M}_\alpha = \{ M \subseteq \omega :$ THE SET $\{ C \in \mathcal{C}_\alpha : |C \cap M| = \omega \}$ HAS SIZE $2^\omega \}$.

HAVING $|\mathfrak{M}_\alpha| \le 2^\omega$, APPLY BERNSTEIN-SIERPIŃSKI-KURATOWSKI AND ASSIGN TO EACH $M \in \mathfrak{M}_\alpha$ SOME $C(M) \in \mathcal{C}_\alpha$ WITH $M \cap C(M)$ INFINITE AND WITH $C(M) \neq C(M')$ FOR DISTINCT $M, M' \in \mathfrak{M}_\alpha$.

LET $\mathcal{A}_\alpha$ BE THE COLLECTION OF ALL $M \cap C(M)$ FOR $M \in \mathfrak{M}_\alpha$, ALL INFINITE $C(M) \setminus M$ FOR $M \in \mathfrak{M}_\alpha$, AND OF ALL $C \in \mathcal{C}_\alpha \setminus \{ C(M) : M \in \mathfrak{M}_\alpha \}$.

CLEARLY, THE RESULTING MATRIX $\{\mathcal{A}_\alpha : \alpha < \mathfrak{h}\}$ IS SHATTERING AND REFINING. IF $M \in [\omega]^\omega$, THEN BY OBSERVATION 6, FOR SOME $\alpha < \mathfrak{h}$ WE HAVE

$$|\{B \in \mathcal{B}_\alpha : |M \cap B| = \omega\}| = 2^\omega.$$

CLEARLY, FOR THIS α, $M \in \mathcal{M}_\alpha$ AND HENCE $M \ni C(M) \cap M \in \mathcal{A}_\alpha$. $\square$

OBSERVATION 8. LET $\mathcal{M} \subseteq [\omega]^\omega$ BE A FAMILY OF SIZE $< 2^\omega$. THEN THERE IS A MAD FAMILY $\mathcal{A}$ SUCH THAT EACH $M \in \mathcal{M}$ MEETS AT LEAST 2 MEMBERS OF $\mathcal{A}$ IN AN INFINITE INTERSECTION.

PROOF. LET $\mathcal{A}$ BE A MAXIMAL AND ALMOST DISJOINT SUBFAMILY OF

$\{A \subseteq \omega : |A| = \omega \ \& \ (\forall M \in \mathfrak{M})\, M \nsubseteq^* A\}$.

$\mathcal{A}$ IS AS REQUIRED. INDEED, IF $X \in [\omega]^\omega$, THEN THERE IS AN ALMOST DISJOINT FAMILY $\mathcal{C}$ ON X OF SIZE 2^ω. SINCE $|\mathfrak{M}| < 2^\omega$, THERE MUST EXIST SOME $C \in \mathcal{C}$ SUCH THAT NO $M \in \mathfrak{M}$ SATISFIES $M \subseteq^* C$. THUS $\mathcal{A}$ IS MAD.

GIVEN $M \in \mathfrak{M}$, THERE IS SOME $A \in \mathcal{A}$ WITH $|A \cap M| = \omega$ BY MAXIMALITY OF $\mathcal{A}$. BUT $|M \setminus A| = \omega$ AND SO THERE IS ALSO $A' \in \mathcal{A}$, $A' \neq A$ WITH $|A' \cap M| = \omega$. $\square$

OBSERVATION 9. $\mathfrak{h}_\omega \leq cf(2^\omega)$.

PROOF. EXPRESS

$$[\omega]^\omega = \bigcup\{\mathfrak{M}_\alpha : \alpha < cf(2^\omega)\}$$

SUCH THAT FOR EACH α, $|\mathfrak{M}_\alpha| < 2^\omega$. APPLY OBSERVATION 8 TO GET $\mathcal{A}_\alpha$.

THE MATRIX $\{\mathcal{A}_\alpha : \alpha < \mathrm{cf}(2^\omega)\}$ IS SHATTERING. $\square$

OBSERVATION 10. $\mathfrak{h} \leq \mathfrak{s}$.

PROOF. COMPARE THE DEFINITIONS:

$\mathfrak{h} = \min\{|\mathcal{H}| : \mathcal{H}$ IS A SHATTERING MATRIX$\}$

$\mathfrak{s} = \min\{|\mathcal{H}| : \mathcal{H}$ IS A SHATTERING MATRIX, CONSISTING OF MAD FAMILIES OF SIZE 2$\}$. $\square$

OBSERVATION 11. $\mathfrak{h} \leq \mathfrak{b}$.

PROOF. FOR $X \in [\omega]^\omega$, LET e_X BE ITS ENUMERATION FUNCTION; I.E. e_X IS A STRICTLY INCREASING MAPPING FROM ω ONTO X.

FIX A FAMILY $\{f_\alpha : \alpha < \mathfrak{b}\} \subseteq {}^\omega\omega$ UNBOUNDED IN $({}^\omega\omega, \leq^*)$.

PUT $\mathcal{A}_\alpha$ TO BE A MAD FAMILY OF SETS A WITH $f_\alpha < e_A$.

THE MATRIX $\{\mathcal{A}_\alpha : \alpha < \mathfrak{b}\}$ IS CLEARLY SHATTERING. $\square$

OBSERVATION 12. $\mathfrak{t} \leq \mathfrak{h}$.

PROOF. CHOOSE A SHATTERING AND REFINING MATRIX $\{\mathcal{A}_\alpha : \alpha < \mathfrak{h}\}$ AND CONSIDER ANY MAXIMAL FAMILY $\mathcal{C} \subseteq \bigcup\{\mathcal{A}_\alpha : \alpha < \mathfrak{h}\}$ LINEARLY ORDERED BY $\subseteq^*$. THEN $|\mathcal{C}| \leq \mathfrak{h}$ BECAUSE $|\mathcal{C} \cap \mathcal{A}_\alpha| \leq 1$ AND $\mathcal{C}$ IS A NOWHERE DENSE TOWER, HENCE $\mathfrak{t} \leq |\mathcal{C}|$. $\square$

DEFINITION. LET $\mathbb{B}$ BE A BOOLEAN ALGEBRA, κ, λ, μ CARDINALS. $\mathbb{B}$ IS (κ, λ, μ)-DISTRIBUTIVE, IF FOR EACH FAMILY $\{P_\alpha : \alpha < \kappa\}$ OF PARTITIONS OF UNITY SUCH THAT $(\forall \alpha < \kappa)\ |P_\alpha| \leq \lambda$, THERE IS

A PARTITION OF UNITY, Q, WITH THE PROPERTY THAT $(\forall q \in Q)(\forall \alpha < \kappa)$

$$|\{p \in P_\alpha : p \wedge q > \mathbb{0}\}| < \mu.$$

A BOOLEAN ALGEBRA $\mathbb{B}$ IS NOWHERE (κ, λ, μ)-DISTRIBUTIVE, IF FOR EACH $b \in \mathbb{B}^+$, $\mathbb{B} \upharpoonright b$ IS NOT (κ, λ, μ)-DISTRIBUTIVE.

IN THE CASE WHEN THERE IS NO RESTRICTION ON THE SIZE OF P_α'S, WE SHALL SPEAK ABOUT $(\kappa, \cdot, \mu)$-DISTRIBUTIVITY OR NOWHERE $(\kappa, \cdot, \mu)$-DISTRIBUTIVITY.

$$\mathfrak{h} = \min\{\kappa : \mathcal{P}(\omega)/_{fin} \text{ IS NOT } (\kappa, \cdot, 2)\text{-DISTRIBUTIVE}\}$$

THEOREM (BASE TREE) [BALCAR, PELANT, S

(A) $\mathfrak{h} = \min\{\kappa : \mathcal{P}(\omega)/\mathit{fin}$ IS NOT $(\kappa, \cdot, 2^{\omega})$-DISTRIBUTIVE$\}$.

(B) THERE EXISTS A FAMILY $T \subseteq \mathcal{P}(\omega)/\mathit{fin}$ SUCH THAT

(i) T IS A DENSE SUBSET OF $\mathcal{P}(\omega)/\mathit{fin}$

(ii) $\langle T, \geqslant \rangle$ IS A TREE OF HEIGHT $\mathfrak{h}$

(iii) EACH LEVEL T_α IS A PARTITION OF UNITY

(iv) EACH $t \in T$ HAS 2^{ω} IMMEDIATE SUCCESSORS.

PROOF. $\bigcup_{n=1}^{8}$ OBSERVATION n. $\square$

A TREE T AS IN (B) – A BASE TREE

THEOREM [BALCAR, VOJTÁŠ]. LET $\{R_n : n \in \omega\}$ BE A PARTITION OF ω. THEN THE FAMILY

$$\mathcal{M} = \{M \subseteq \omega : \limsup_{n\in\omega} |M \cap R_n| = \infty\}$$

HAS AN ALMOST DISJOINT REFINEMENT.

PROOF. FOR $M \in \mathcal{M}$, LET

$$dom(M) = \{n \in \omega : M \cap R_n \neq \emptyset\}.$$

WE MAY AND SHALL ASSUME THAT FOR EACH $M \in \mathcal{M}$ AND ANY TWO $n < k$, $n, k \in dom(M)$,

$$|M \cap R_n| < |M \cap R_k| < \omega.$$

A TRANSVERSAL IS AN INFINITE SUBSET OF ω, WHICH MEETS EACH R_n IN AT MOST 1 POINT. AN ALMOST DISJOINT REFINEMENT

WE ARE LOOKING FOR, WILL CONSIST OF TRANSVERSALS.

FIX A BASE TREE ON ω AND REPRESENT ITS LEVELS AS MAD FAMILIES $\mathcal{A}_\alpha$ FOR $\alpha < \mathfrak{h}$.

TRASFINITE INDUCTION TO $\mathfrak{h}$:

LET $\mathfrak{M}_\alpha = \{M \in \mathfrak{M} : |\{A \in \mathcal{A}_\alpha : A \subseteq^* dom(M)\}| = 2^\omega\}$

FOR EACH $M \in \mathfrak{M}_\alpha$, CHOOSE $A(M) \in \mathcal{A}_\alpha$ SUCH THAT FOR $M \neq M'$, $A(M) \cap A(M')$ IS FINITE.

FOR EACH $M \in \mathfrak{M}_\alpha$, CHOOSE A TRANSVERSAL $T(M) \subseteq M$ SUCH THAT $dom\ T(M) = A(M)$ AND $T(M)$ IS ALMOST DISJOINT WITH ALL ELEMENTS OF THE REFINEMENT CONSTRUCTED UP TO NOW.

THIS IS ALWAYS POSSIBLE!

INDEED, IF A AND A' ARE DISTINCT MEMBERS OF $\bigcup_{\beta \le \alpha} \mathcal{A}_\beta$ AND $A = \operatorname{dom} T(M)$, $A' = \operatorname{dom} T(M')$ AND A, A' ARE ALMOST DISJOINT, THEN $T(M)$ AND $T(M')$ ARE ALMOST DISJOINT AS WELL.

AND, A AND A' ARE NOT ALMOST DISJOINT FOR $A \in \mathcal{A}_\alpha$, IF $A' \in \mathcal{A}_\beta$ FOR SOME $\beta < \alpha$, AND IN SUCH A CASE SUCH A' IS UNIQUE AND SATISFIES $A \subseteq^* A'$.

THEREFORE, THE NUMBER OF TRANSVERSALS, WHICH MEET THE SET $\bigcup \{ M \cap R_n : n \in A \}$ IN AN INFINITE INTERSECTION, IS OF SIZE AT MOST $|\alpha| < \mathfrak{h} \le \mathfrak{b} \le \mathfrak{a}$.

SO THERE IS A ROOM FOR ANOTHER ALMOST DISJOINT $T(M)$. $\square$

COROLLARY. THE FAMILY OF ALL SUBSETS OF ω WHICH HAVE A POSITIVE UPPER BANACH DENSITY HAS AN ALMOST DISJOINT REFINEMENT.

A SET $A \subseteq \omega$ HAS A POSITIVE UPPER BANACH DENSITY IF THERE IS A SEQUENCE OF INTERVALS $\langle I_n : n \in \omega \rangle$ OF INCREASING LENGTHS SUCH THAT

$$\limsup_{n \in \omega} \frac{|I_n \cap A|}{|I_n|} > 0$$

LET $\{R_n : n \in \omega\}$ BE DEFINED BY $R_n = [n^n, (n+1)^{n+1})$.

[SZEMERÉDI 1975] EVERY SET OF POSITIVE UPPER BANACH DENSITY CONTAINS ARBITRARILY LONG FINITE ARITHMETIC PROGRESSIONS.

SO, IF THE SET M HAS A POSITIVE UPPER BANACH DENSITY, THEN

$$\limsup_{m\in\omega} |M \cap R_x| = \infty.$$

THE THEOREM APPLIES.

COROLLARY. THE FAMILY OF ALL SUBSETS OF $\mathbb{R}$ WHICH HAVE INFINITELY MANY ACCUMULATION POINTS HAS AN ALMOST DISJOINT REFINEMENT.

— NOTICE THAT THE PROOF OF BALCAR-VOJTÁŠ THEOREM DID NOT MAKE ANY USE FROM THE FACT THAT R_x'S ARE COUNTABLE. WHAT WAS REALLY NEEDED WAS THE FACT THAT $|\mathcal{M}| \le 2^{\omega}$.

GIVEN A REAL x, THEN THE

FAMILY OF ALL COUNTABLE SUBSETS OF $\mathbb{R}$ WITH x AS A POINT FROM THE SECOND DERIVED SET, HAS AN ALMOST DISJOINT REFINEMENT BY SEQUENCES CONVERGING TO x. INDEED, CONSIDER

$$R_n = \{ t \in \mathbb{R} : 2^{-n-1} < |x-t| \leq 2^{-n} \}.$$

AND SEQUENCES CONVERGING TO x ARE ALMOST DISJOINT FROM SEQUENCES CONVERGING TO y, FOR $x \neq y$.

FINALLY, FOR THOSE SETS, WHICH HAVE INFINITELY MANY ACCUMULATION POINTS, BUT AN EMPTY SECOND DERIVED SET, CHOOSE

$$R_n = [-n-1, -n) \cup [n, n+1).$$

COROLLARY. LET $\kappa \leq 2^\omega$. THE FAMILY $\mathfrak{M} = \{M \subseteq \kappa : \text{ORDER TYPE}(M) = \omega^2\}$ HAS AN ALMOST DISJOINT REFINEMENT.

– SIMILARLY AS BEFORE, DO IT SEPARATELY FOR EACH $\alpha < \kappa$, WHICH IS A LIMIT OF COUNTABLY MANY LIMIT ORDINALS.

PROOF OF BALCAR-VOJTÁŠ THEOREM.

LET $\mathcal{U}$ BE A UNIFORM ULTRAFILTER ON ω.

LET τ BE THE LENGTH OF A MAXIMAL $\supseteq^*$-DECREASING SUBSET OF $\mathcal{U}$.

WE HAVE A FAMILY $\{U_\alpha : \alpha < \tau\} \subseteq \mathcal{U}$ SATISFYING, BY MAXIMALITY, THAT

$(\forall U \in \mathcal{U})(\exists \alpha < \tau)\ |U \setminus U_\alpha| = \omega$.

WE CAN ASSUME THAT FOR EACH $\alpha < \beta < \tau$, $|U_\alpha \setminus U_\beta| = \omega$.

CASE $\tau = \omega$: PUT $R_n = \bigcap_{i<n} U_i \setminus U_n$,

WE HAVE A PARTITION $\{R_n : n \in \omega\}$ SUCH THAT FOR EACH $U \in \mathcal{U}$,

$\{n \in \omega : |U \cap R_n| = \omega\}$ IS INFINITE.

APPLY PREVIOUS THEOREM.

CASE $\tau > \omega$: FOR $\alpha < \tau$ WITH $cf(\alpha) = \omega$, CHOOSE AN INCREASING SEQUENCE $\langle \alpha_n : n < \omega \rangle$, COFINAL IN α. PUT $R^\alpha_n = \bigcap_{i<n} U_{\alpha_i} \setminus (U_{\alpha_n} \cup U_\alpha)$.

$\{R^\alpha_n : n < \omega\}$ IS A PARTITION OF $\omega \setminus U_\alpha$; APPLY PREVIOUS THEOREM TO GET AN ALMOST DISJOINT FAMILY CONSISTING OF TRANSVERSALS CONTAINED IN $\omega \setminus U_\alpha$; CALL IT $\mathcal{T}_\alpha$.

$\mathcal{T}_\alpha$ IS ALMOST DISJOINT. IF $\alpha < \beta < \tau$ ARE TWO ORDINALS OF COUNTABLE COFINALITY, $T \in \mathcal{T}_\alpha$ AND $T' \in \mathcal{T}_\beta$, THEN $T' \subseteq^* U_\alpha$ AND $T \cap U_\alpha = \emptyset$, SO T AND T' ARE ALMOST DISJOINT.

LET $\mathcal{T} = \bigcup \{\mathcal{T}_\alpha : \alpha < \tau,\ cf(\alpha) = \omega\}$.

$\mathcal{T}$ IS THE REQUIRED ALMOST DISJOINT

REFINEMENT OF $\mathcal{U}$: INDEED, IF $U \in \mathcal{U}$ IS ARBITRARY, THEN THERE IS SOME $\alpha_0 < \tau$ WITH $U \setminus U_{\alpha_0}$ INFINITE. FOR $U \cap U_{\alpha_0}$, THERE IS SOME $\alpha_1 < \tau$ WITH $U \cap U_{\alpha_0} \setminus U_{\alpha_1}$ INFINITE. PROCEEDING FURTHER, WE GET A STRICTLY INCREASING SEQUENCE $\langle \alpha_n : n < \omega \rangle$, LET α BE ITS SUPREMUM. NOW, $U \cap R^{\alpha}_{n}$ IS INFINITE FOR INFINITELY MANY n'S, SO THERE IS SOME $T \in \mathcal{T}_\alpha \subseteq \mathcal{T}$ WITH $T \subseteq U$. $\square$

ORIGINS OF THE PROBLEM.

A TOPOLOGICAL SPACE ω^* IS NOT EXTREMALLY DISCONECTED. THEREFORE THERE IS A POINT $p \in \omega^*$ AND TWO DISJOINT OPEN SETS U, V SUCH THAT $p \in \overline{U} \cap \overline{V}$.

DEFINITION. LET X BE A TOPOLOGICAL SPACE, $p \in X$, κ A CARDINAL NUMBER. A POINT p IS CALLED A κ-POINT, IF THERE IS A PAIRWISE DISJOINT FAMILY $\mathcal{V}$ OF OPEN SETS, $|\mathcal{V}| = \kappa$, SUCH THAT $p \in \overline{V}$ FOR EACH $V \in \mathcal{V}$.

– REPLACING POINT p BY SET $Z \subseteq X$ AND REQUIRING $Z \subseteq \overline{V}$ IN THE ABOVE, THE SET Z IS A κ-SET.

QUESTION: [R.S. PIERCE, 1967]

IS THERE A 3-POINT IN ω^*?

USING STANDARD IDENTIFICATION

$\omega^* = \beta\omega \setminus \omega = St(\mathcal{P}(\omega)/fin)$

WE HAVE POINTS IN ω^* IDENTIFIED WITH FREE ULTRAFILTERS ON ω.

FOR $A \subseteq \omega$, $A^* = cl_{\beta\omega}(A) \cap \omega^*$.

THEOREM: THE FOLLOWING ARE EQUIVALENT FOR A SET $Z \subseteq \omega^*$:

(i) Z IS A 2^ω-SET

(ii) A FAMILY $\mathcal{M} = \{M \subseteq \omega : M^* \cap Z \neq \emptyset\}$ HAS AN ALMOST DISJOINT REFINEMENT.

PROOF. (i) $\to$ (ii): LET $\mathcal{V}$ BE A SET OF PAIRWISE DISJOINT OPEN SUBSETS OF ω^* WITH $|\mathcal{V}| = 2^\omega$ AND $Z \subseteq \overline{V}$ FOR EACH $V \in \mathcal{V}$. SINCE $|\mathcal{M}| \leq 2^\omega$, CHOOSE FOR EACH $M \in \mathcal{M}$ SOME $V(M) \in \mathcal{V}$ WITH $V(M) \neq V(M')$ FOR DISTINCT $M, M' \in \mathcal{M}$.

FOR EACH $M \in \mathcal{M}$ AND EACH $V \in \mathcal{V}$ WE HAVE THAT THE SET $M^* \cap V$ IS A NON-EMPTY OPEN SUBSET IN ω^*. INDEED, $M^* \cap Z \neq \emptyset$ AND $Z \subseteq \overline{V}$, SO $M^* \cap V \neq \emptyset$.

SO WE CAN FOR EACH $M \in \mathcal{M}$ CHOOSE AN INFINITE SET $A(M) \subseteq M$ WITH $A(M)^* \subseteq M^* \cap V(M)$. THE FAMILY $\{A(M) : M \in \mathcal{M}\}$ IS APPARENTLY A REFINEMENT OF $\mathcal{M}$ AND IS ALSO ALMOST DISJOINT, FOR IF $M \neq M'$, THEN $A(M)^* \subseteq V(M)$ AND $A(M')^* \subseteq V(M')$ AND $V(M), V(M')$ WERE CHOSEN DISTINCT, HENCE DISJOINT.

(ii) $\rightarrow$ (i): LET $\mathcal{A}$ BE AN ALMOST DISJOINT REFINEMENT OF A FAMILY $\mathcal{M}$. FOR EACH $A \in \mathcal{A}$, FIX AN

ALMOST DISJOINT FAMILY $\{A_\alpha : \alpha < 2^\omega\}$ OF SUBSETS OF A. PUT

$$V_\alpha = \bigcup \{A_\alpha^* : A \in \mathcal{A}\}.$$

AS A UNION OF CLOPEN SUBSETS OF ω^*, EACH V_α IS OPEN.

IF $\alpha \neq \beta$, THEN $V_\alpha \cap V_\beta = \emptyset$: THE SETS A_α, A'_β ARE ALMOST DISJOINT, SINCE $\mathcal{A}$ IS ALMOST DISJOINT AS WELL AS EACH $\{A_\alpha : \alpha < 2^\omega\}$.

LET $\alpha < 2^\omega$ BE ARBITRARY, LET $\mathcal{U} \in Z$ BE ARBITRARY AND LET $M \in \mathcal{U}$ BE ARBITRARY. THEN THERE IS SOME $A \subseteq \mathcal{A}$ WITH $A \subseteq M$. SO ALSO $A_\alpha \subseteq M$. THUS $A_\alpha^* \subseteq M^*$ AND SO $M^* \cap V_\alpha \neq \emptyset$. CONSEQUENTLY $\mathcal{U} \in \overline{V_\alpha}$ AND FINALLY $Z \subseteq \overline{V_\alpha}$. $\square$