Uniwersytet Wrocławski

Marczewski-like ideals related to superperfect trees.

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February 2024.
Hejnice.

Winter School in Abstract Analysis 2024
section Set Theory & Topology
(Joint work with Aleksander Cieślak)

Basic Definitions

Definition

A set $A \subseteq \omega^\omega$ is nowhere dense if for every $U \in \mathcal{O}$ there is $V \in \mathcal{O}, V \subseteq U$ such that $V \cap A = \emptyset$.

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Definition

A set $A \subseteq \omega^\omega$ is $\mathbb{S}$-<u>small</u> if for every $T \in \mathbb{S}$ there is $S \in \mathbb{S}$ such that $S \subseteq T$ and $[T] \cap A = \emptyset$.

$\mathbb{S}$ is the collection of perfect trees on $2^{<\omega}$.

Definition

The <u>Marczewski ideal</u> s_0 is the collection of $\mathbb{S}$-small sets.

All Borel $\mathbb{S}$-small sets are countable!

Remark

There is an uncountable $\mathbb{S}$-small set.

Proof: If CH holds take a Luzin set. If not take any set of size $\aleph_1$.

[1] Combinatorial forcing trees

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Theorem (Marczewski 1935, Fremlin)

The ideal of $\mathbb{S}$-small sets (called the Marczewski ideal) does not have a Borel basis. In fact $\mathrm{cof}(s_0) > \mathfrak{c}$.

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Definition

Given a forcing notion of trees $\mathbb{T}$[1] on $\omega^{<\omega}$ (Sacks, Miller, Laver, ...), a set $X \subset \omega^\omega$ is $\mathbb{T}$-small if for every $T \in \mathbb{T}$ there is $S \subseteq T, S \in \mathbb{T}$ such that $[S] \cap X = \emptyset$. The ideal of $\mathbb{T}$-small sets will be denoted by t_0.

[1] Combinatorial forcing trees

Generalized Marczewski ideals

What can be said about the ideal m_0, the ideal of small sets according to the Miller/Superperfect tree forcing?

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Theorem (Brendle, Khomskii, Wohofsky, 2016)

If $\mathbb{T}$ is a tree forcing notion, then $\mathrm{cof}(t_0) > 0$ if one of the following hold.

- $\mathbb{T}$ has the Constant or 1-1 property: For every $T \in \mathbb{T}$ and every continuous $f : [T] \to \omega$ there is $S \in \mathbb{T}$ such that $S \subseteq T$ and $f|[S]$ is constant or 1-1.
- $\mathbb{T}$ has the Incompatibility shrinking property: If $\kappa < \mathfrak{c}$ and $\{A_\alpha : \alpha < \kappa\} \subseteq t_0^B$, then there is $T \in \mathbb{T}$ such that $A_\alpha \cap [T] = \emptyset$ for every $\alpha < \kappa$.

Theorem (Brendle, Khomskii, Wohofsky, 2016)

Under $\mathfrak{d} = \mathfrak{c}$, $\mathbb{M}$ has the incompatibility shrinking property.

Theorem (Miller)

$\mathbb{M}$ has the constant or 1-1 property.

Therefore $\mathrm{cof}(m_0) > \mathfrak{c}$

Miller of an ideal

Definition

Given an ideal $\mathscr{I}$ on ω, the Miller forcing of $\mathscr{I}$ positive sets, denoted by $\mathbb{M}_{\mathscr{I}}$ are trees $T \subseteq \omega^{<\omega}$ such that for every $\sigma \in T$ there is $\tau \in T$, $\sigma \subseteq \tau$ such that $succ_T(\tau)$ is not in $\mathcal{I}$.

The collection of $\mathbb{M}_{\mathscr{I}}$ small sets will be denoted by $m(\mathscr{I})_0$ and its Borel part will be $m(\mathscr{I})_0^B$.

Remark

$\mathbb{M}_{\mathrm{Fin}}$ is the regular Miller forcing.

Questions

For which ideals $\mathscr{I}$ does $\mathbb{M}_{\mathscr{I}}$ has the constant or 1-1 property? In which cases $\mathbb{M}_{\mathscr{I}}$ has the incompatibility shrinking property?

Constant or 1-1 property

Theorem (Sabok, Zapletal, 2011)

$\mathbb{M}_{\mathscr{I}}$ does not have the Constant or 1-1 property if and only if $\mathbb{M}_{\mathscr{I}}$ adds Cohen reals if and only if there is $X \notin \mathscr{I}$ such that $\mathrm{nwd} \leq_K \mathscr{I}|_X$.

Incompatibility shrinking property

Tncompatibility shrinking property as a cardinal invariant:

Definition

The incompatibility shrinking number of $\mathbb{T}$, $\mathrm{is}(\mathbb{T})$ is the smallest number of elements of t_0^B required to hit all $[T]$ for $T \in \mathbb{T}$.

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Remarks

- $\mathbb{T}$ has the incompatibility shrinking property if and only if $\mathrm{is}(\mathbb{T}) = \mathfrak{c}$.
- $\mathrm{add}(t_0^B) \leq \mathrm{is}(\mathbb{T}) \leq \mathrm{cov}(t_0^B)$.

Incompatibility Shrinking Property General Facts

Theorem:

For any ideal $\mathscr{I}$, we have

$$\min\{\mathfrak{b}, \operatorname{cov}(m_0^B(\mathscr{I}))\} \leq \operatorname{is}(\mathbb{M}_{\mathscr{I}}) \leq \operatorname{cov}(m_0^B(\mathscr{I})) \leq \mathfrak{d}.$$

Moreover, if $\mathbb{M}_{\mathscr{I}}$ adds Cohen reals, then

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Ideal $\mathscr{I}$	$\operatorname{is}(\mathbb{M}_{\mathscr{I}}) \in$
Fin	$\{\mathfrak{d}\}$
$\mathrm{conv}, \mathcal{R}, \mathcal{ED}, \mathrm{Fin} \times \mathrm{Fin}$	$[\mathfrak{b}, \mathfrak{d}]$
Analytic p-ideals	$[\operatorname{add}(\mathcal{N}), \mathfrak{d}]$
nwd	$\{\operatorname{add}(\mathcal{M})\}$.

Full Miller forcing

Definition

The forcing $\mathbb{FM}$, the full Miller forcing, is the collection of superperfect trees such that each node splits into either one point or on the whole ω.

$\mathbb{FM}$ does not have the Constant or 1-1 property.

Theorem

$\mathrm{is}(\mathbb{FM}) = \mathrm{add}(\mathcal{M})$.

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~~Miller~~ Laver of an ideal

Definition

Given an ideal $\mathscr{I}$ on ω, the Laver forcing of $\mathscr{I}$ positive sets, denoted by $\mathbb{L}_{\mathscr{I}}$ are trees $T \subseteq \omega^{<\omega}$ such that for every $\sigma \in T$ extending the stem, $succ_T(\sigma)$ is not in $\mathcal{I}$.

The collection of $\mathbb{L}_{\mathscr{I}}$ small sets will be denoted by $\ell(\mathscr{I})_0$ and its Borel part will be $\ell(\mathscr{I})_0^B$.

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$\mathbb{L}_{\mathrm{Fin}}$ is the regular Laver forcing.

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For which ideals $\mathscr{I}$ does $\mathbb{L}_{\mathscr{I}}$ has the constant or 1-1 property? In which cases $\mathbb{L}_{\mathscr{I}}$ has the incompatibility shrinking property?

Constant or 1-1 property

Remark

If $\mathbb{M}_{\mathscr{I}}$ does not have the constant or 1-1 property, then $\mathbb{L}_{\mathscr{I}}$ does not have it either. In particular $\mathbb{L}_{\mathrm{nwd}}$ does not have the constant or 1-1 property.

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Theorem (Gray, 1980)

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Therefore $\mathrm{cof}(\ell_0) > \mathfrak{c}$.

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Theorem

$\mathbb{L}_{\mathscr{I}}$ has the Constant or 1-1 property provided that $\mathscr{I}$ is $p+$ and nowhere maximal.

Question

What can be said about $\mathbb{L}_{\mathrm{Fin}\times\mathrm{Fin}}$ or $\mathbb{L}_{\mathcal{Z}}$?

Incompatibility Shrinking Property

Ideal $\mathscr{I}$	$\mathrm{is}(\mathbb{L}_{\mathscr{I}}) \in$
$\mathrm{Fin}, \mathrm{Fin} \times \mathrm{Fin}$	$\{\mathfrak{b}\}$
$\mathrm{conv}, \mathcal{R}, \mathcal{ED}$	$[\aleph_1, \mathfrak{b}]$
Analytic p-ideals	$[\mathrm{add}(\mathcal{N}), \mathfrak{b}]$
nwd	$\{\mathrm{add}(\mathcal{M})\}.$

Thank you for your attention!

2nd Wrocław Logic Conference.
https://prac.im.pwr.edu.pl/ twowlc/

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References

[1] Aleksander Cieślak and Arturo Martínez-Celis. On ideals related to Laver and Miller trees. 2023. arXiv: *2312.01387 [math.LO]*.