Ramsey theorem for trees with successor operation

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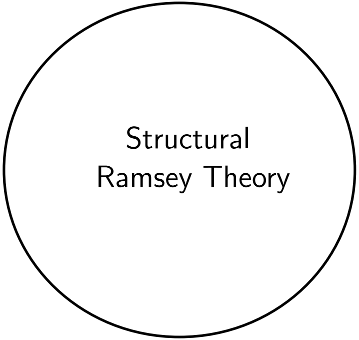
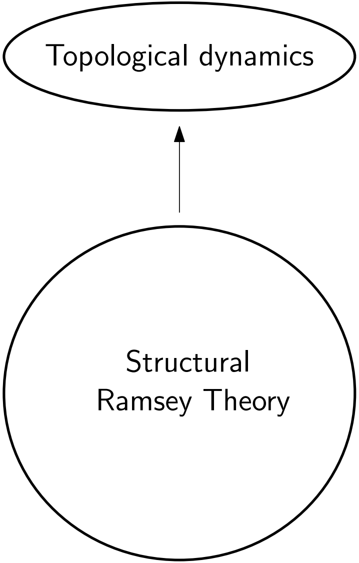
Joint work with Martin Balko, Samuel Bruanfeld, Natasha Dobrinen, David Chodounský, Matěj Konečný, Jaroslav Nešetřil, Noe de Rancourt, Stevo Todorcevic, Lluis Vena, Andy Zucker

Winter school, 2023, Hejnice


Structural
Ramsey Theory


Topological dynamics
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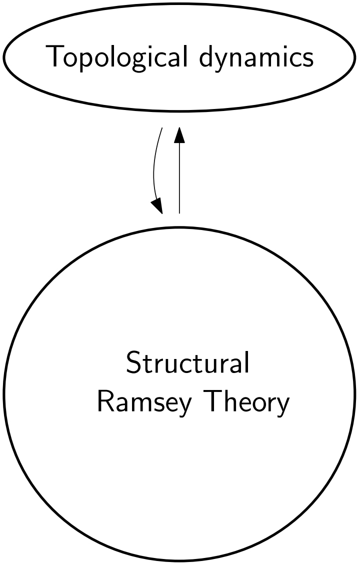
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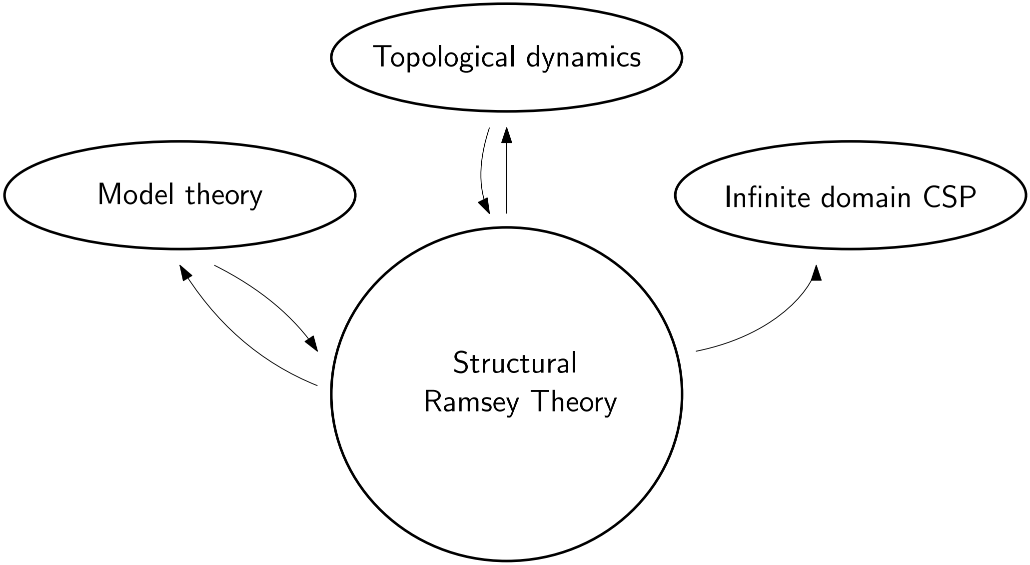
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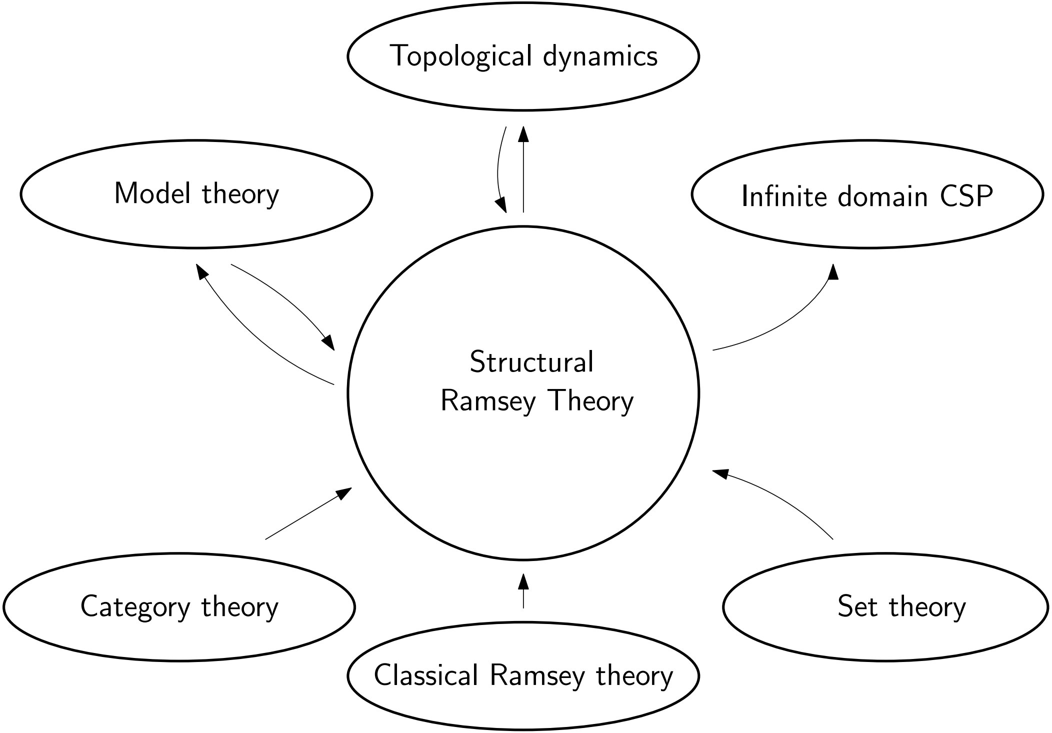
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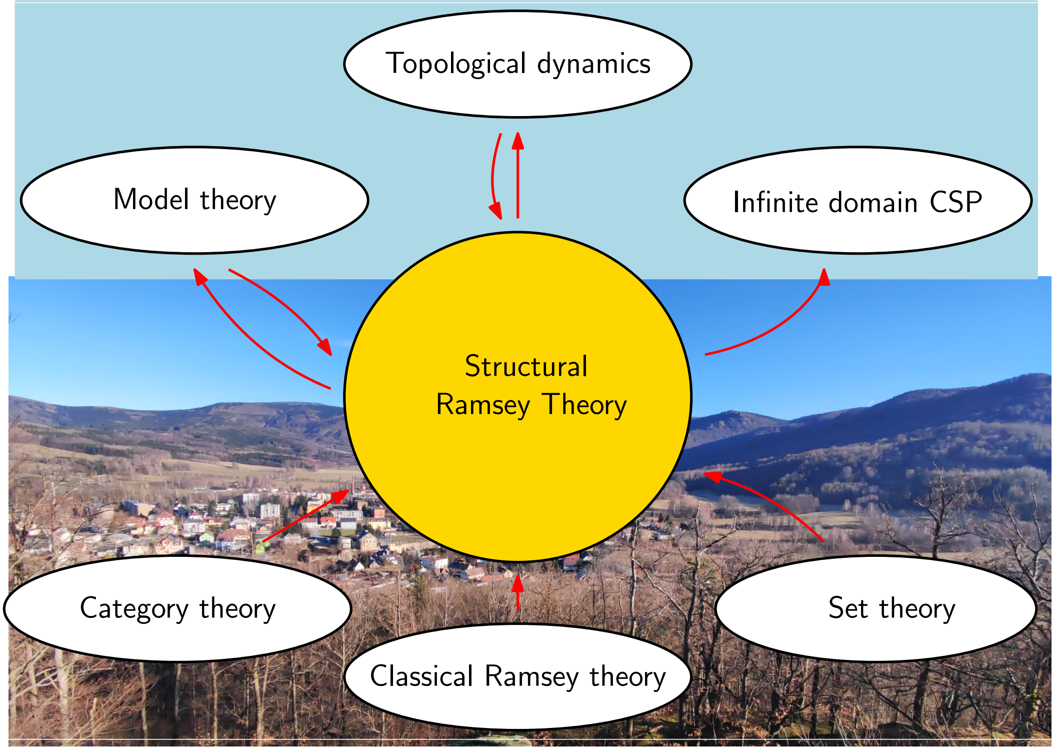
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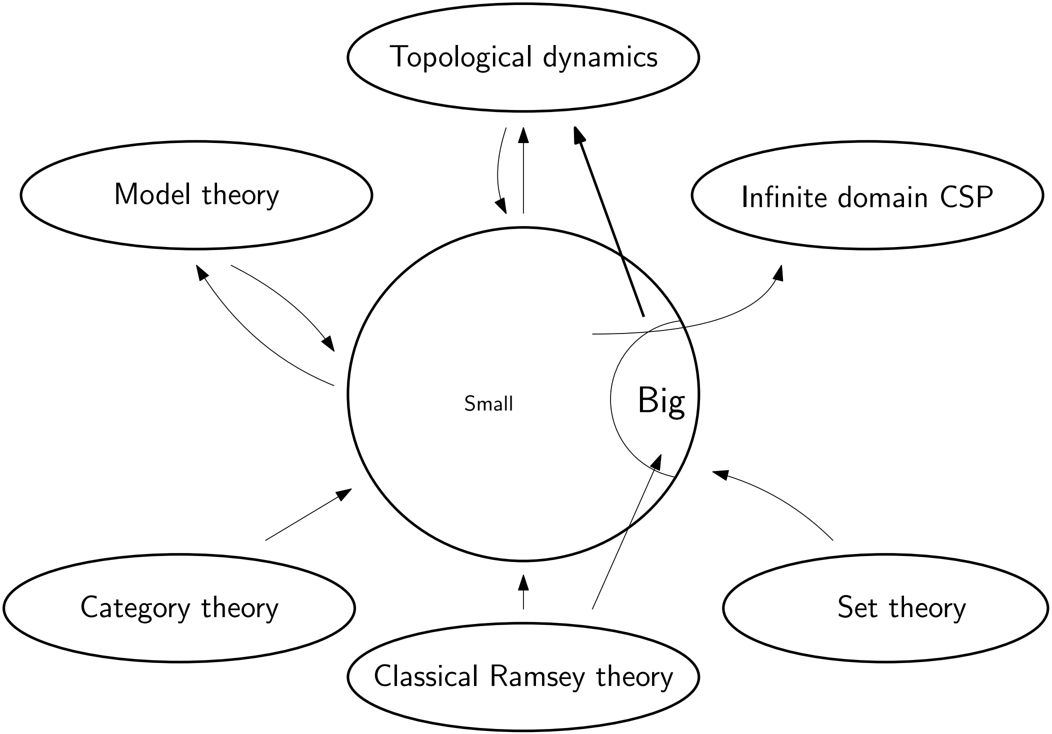

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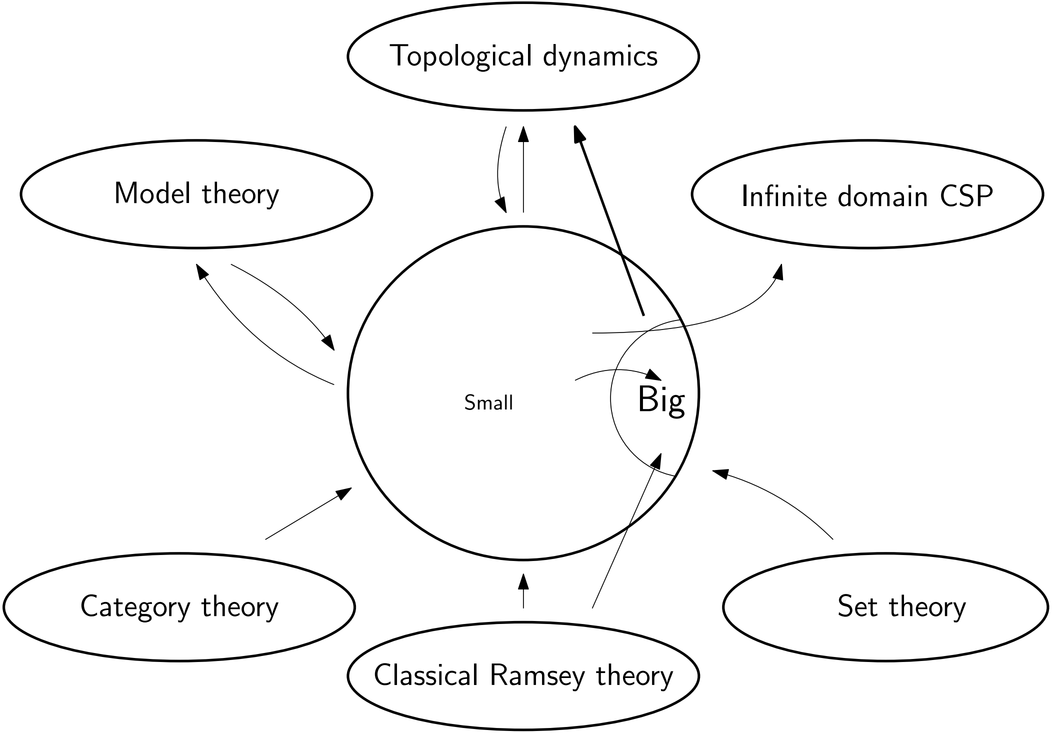
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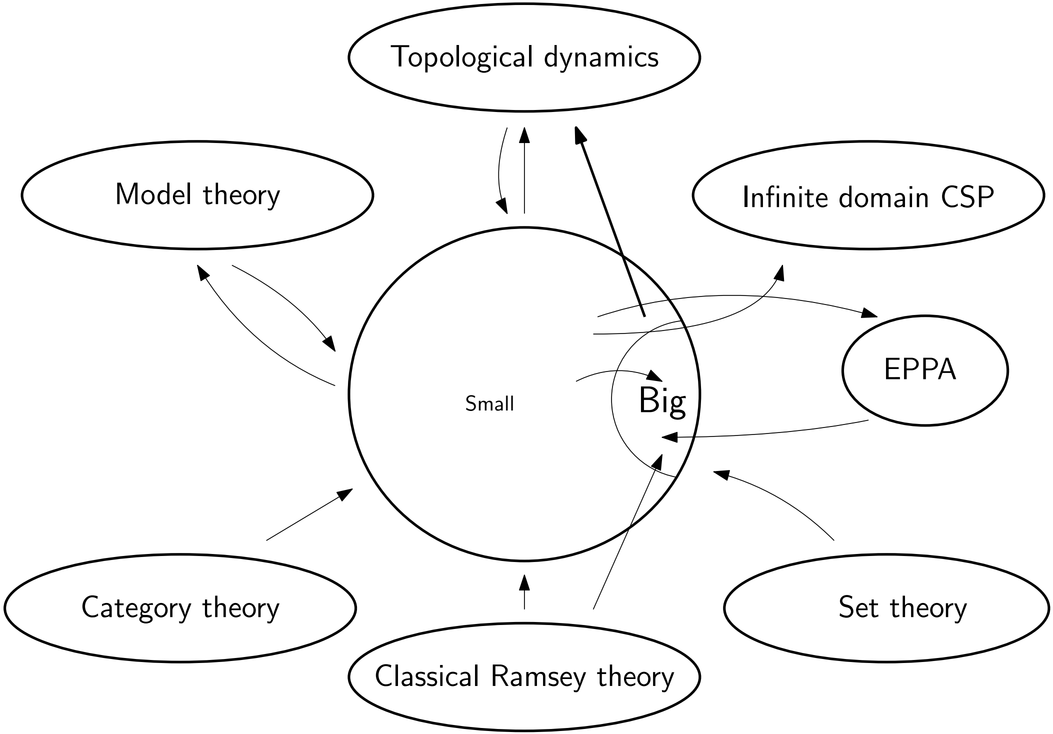
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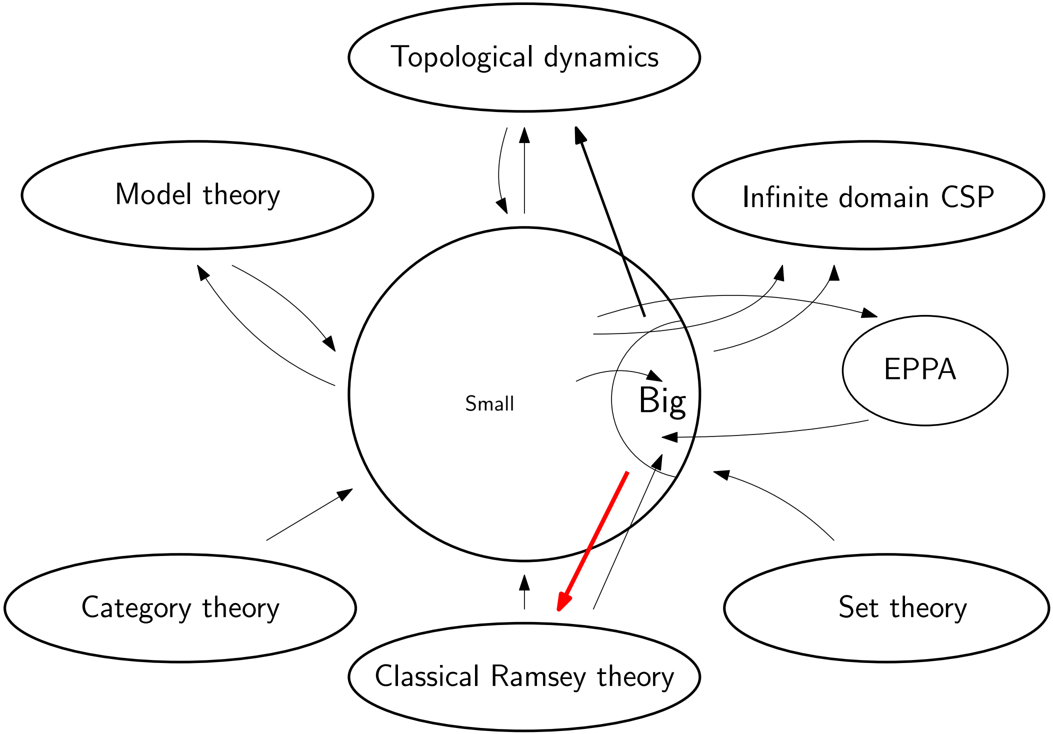

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Big Ramsey Degrees of $(\mathbb{Q}, \leq)$

Theorem (Upper bound by Laver 1969, characterisation by Devlin 1979)

The order of rationals $(\mathbb{Q}, \leq)$ has finite big Ramsey degrees: for every $n \in \omega$ there exists $T(n) \in \omega$ such that whenever n-element subsets of $\mathbb{Q}$ are finitely colored, there exists a copy of $(\mathbb{Q}, \leq)$ in itself touching at most $T(n)$ many colors.

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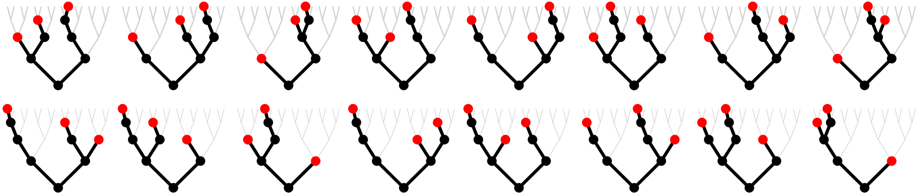

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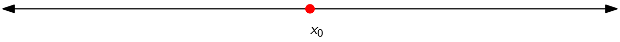

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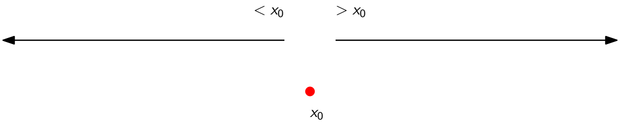

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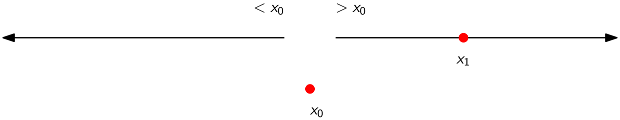

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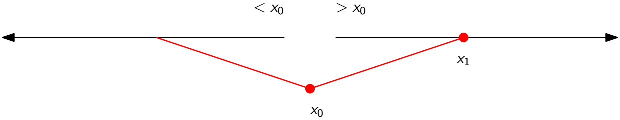

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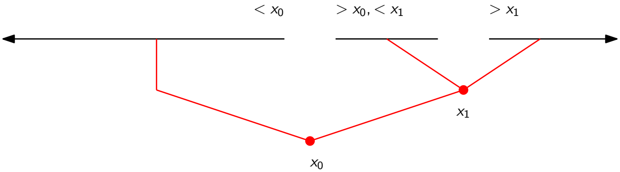

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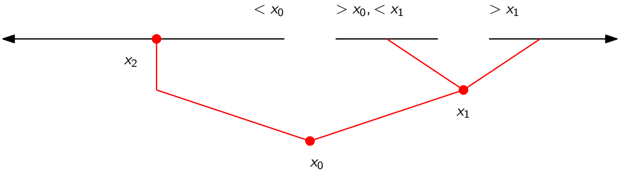

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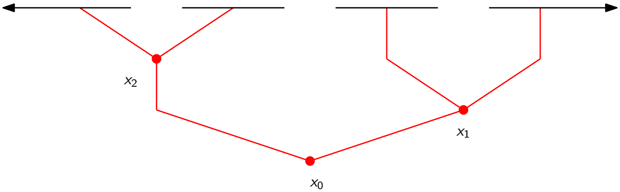

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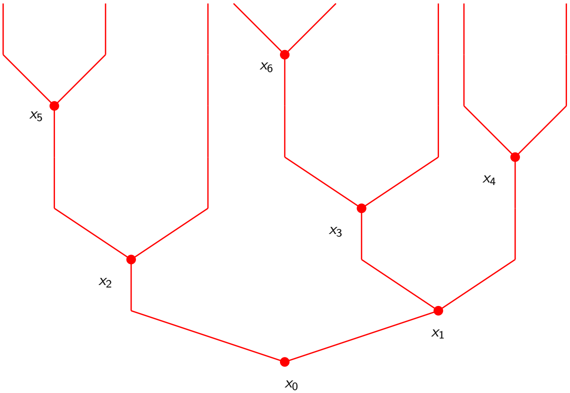

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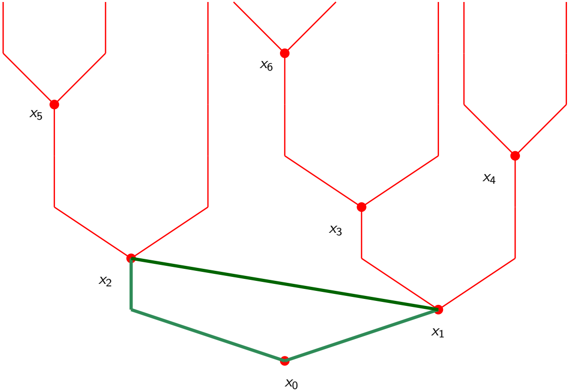

Colour of k-tuple = shape of meet closure in the tree

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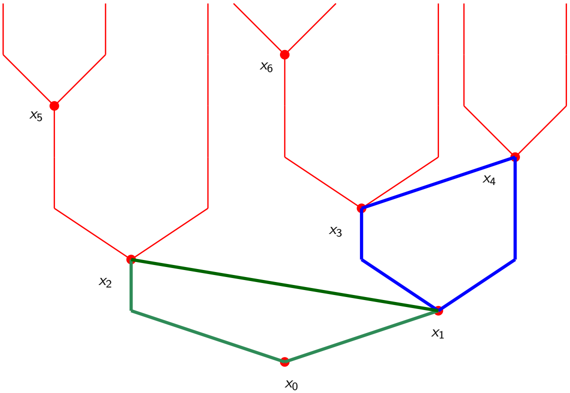

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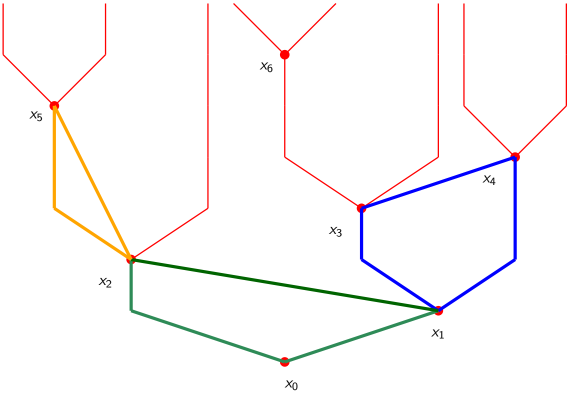

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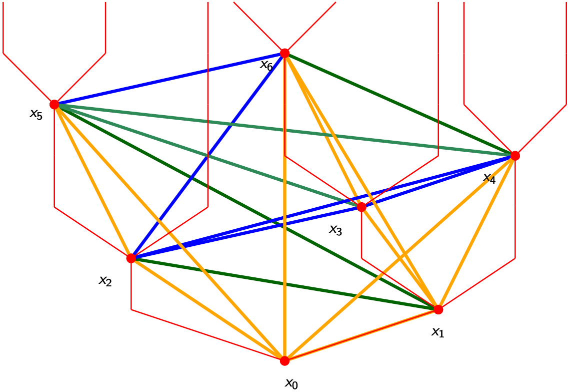

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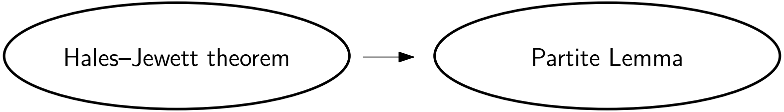
Proof techniques in small Ramsey world are different



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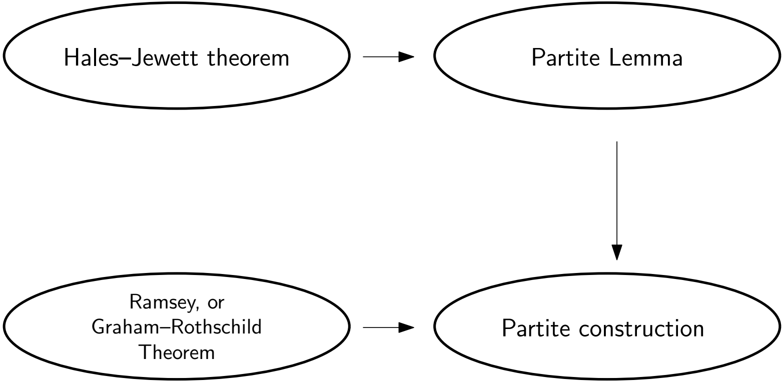

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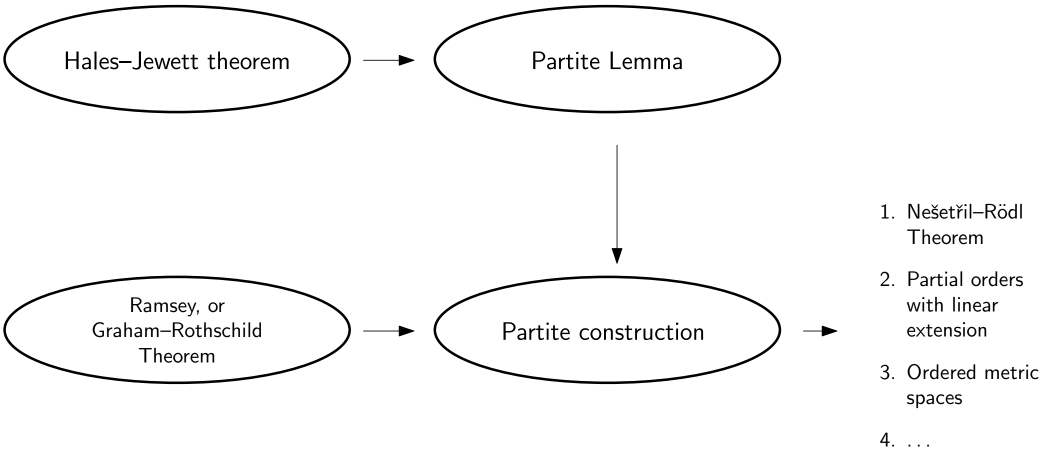

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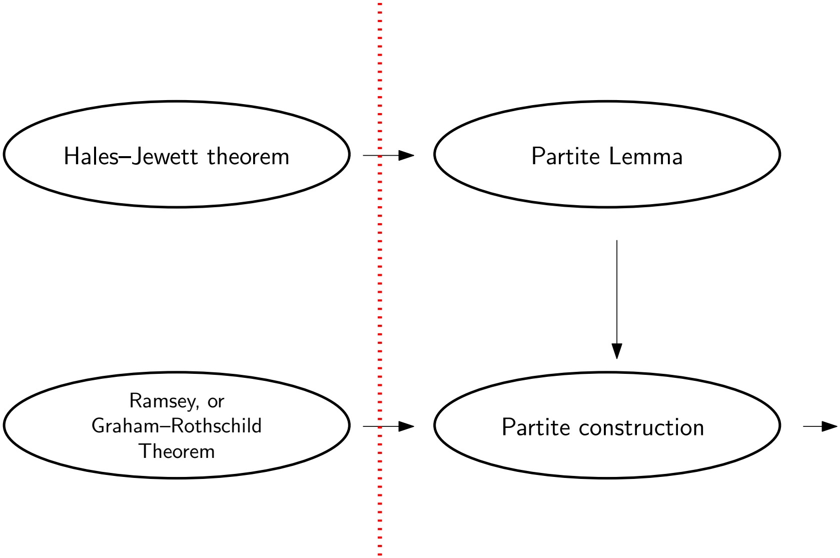

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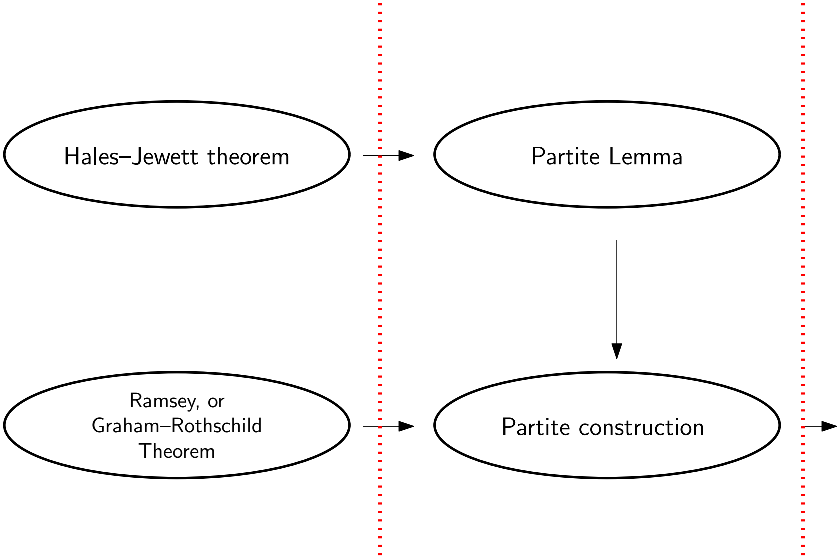

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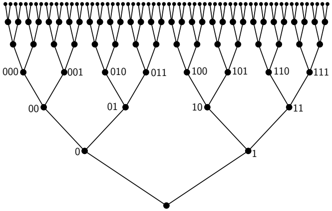

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- A tree is a (possibly empty) partially ordered set $(T, <_T)$ such that, for every $t \in T$, the set $\{ s \in T : s <_T t \}$ is finite and linearly ordered by $<_T$. All trees considered are finite or countable.

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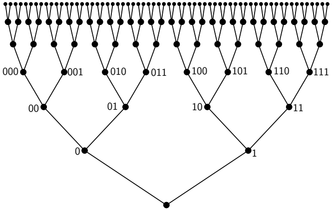

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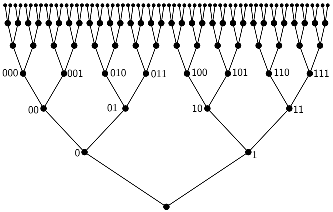

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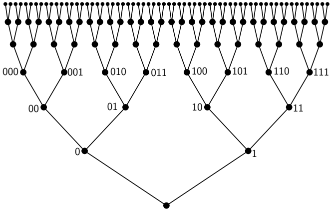

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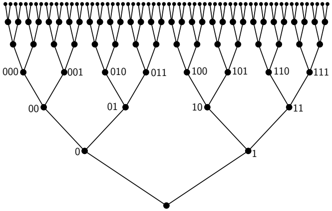

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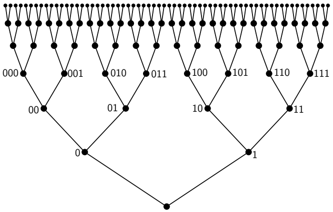

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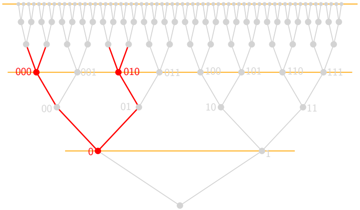
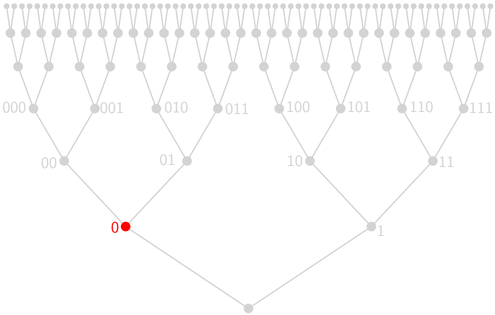
- The height of T, denoted by $h(T)$, is the minimal natural number h such that $T(h) = \emptyset$. If there is no such number h, then we say that the height of T is ω.

Subtrees and strong subtrees



- A subtree of a tree T is a subset $S \subseteq T$ viewed as a tree equipped with the induced partial ordering.
- Given a tree T and nodes $s, t \in T$ we say that s is a successor of t in T if $t \leq_T s$.
- The node s is an immediate successor of t in T if $t <_T s$ and there is no $s' \in T$ such that $t <_T s' <_T s$.
- Node with no successors is leaf.

Strong subtree



Definition

Let T be rooted tree. Nonempty $\mathbf{S} \subseteq \mathbf{T}$ is a strong subtree of T of height $n \in \omega + 1$ if:

1. S is closed for meets. (In particular, S is rooted.)

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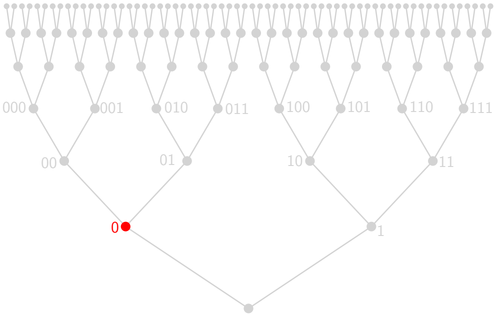

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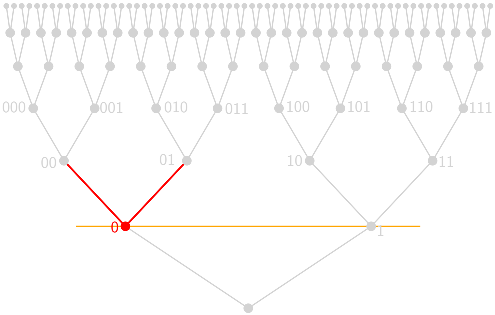

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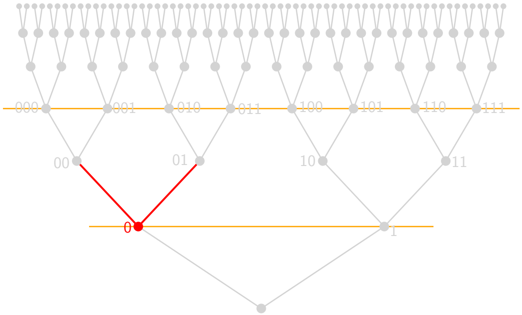

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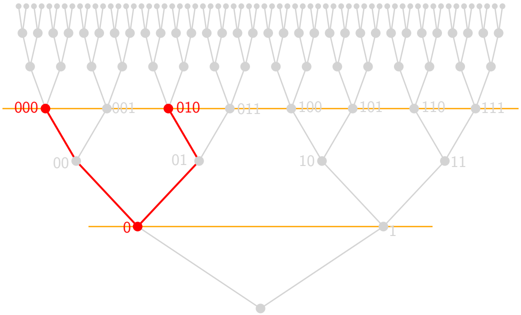

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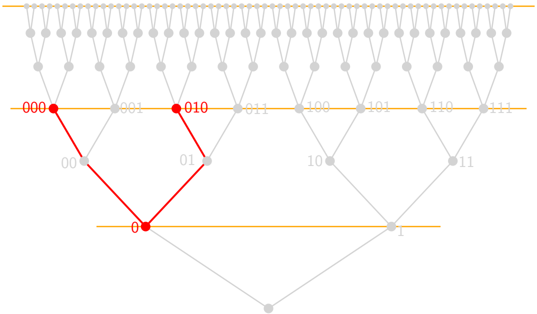

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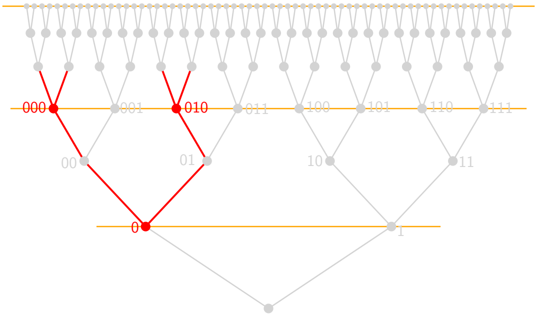

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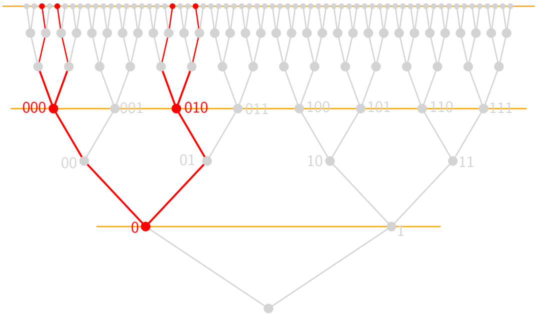

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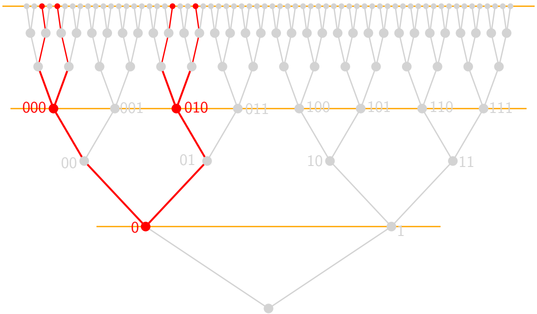
Strong subtree



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Ramsey-type theorem for strong subtrees

Let T be a tree and $k \in \omega + 1$. We use $\mathrm{Str}_k(T)$ to denote the set of all strong subtrees of T of height k.

Theorem (Milliken 1979)

For every rooted finitely branching tree T with no leaves, every $k \in \omega$ and every finite colouring of $\mathrm{Str}_k(T)$ there is $S \in \mathrm{Str}_\omega(T)$ such that the set $\mathrm{Str}_k(S)$ is monochromatic.

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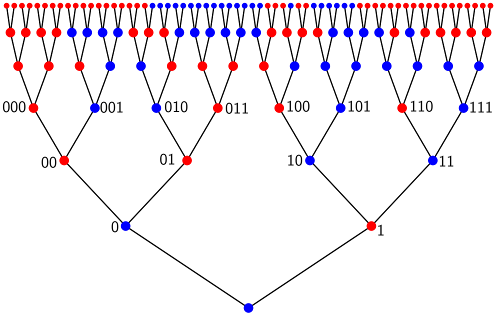

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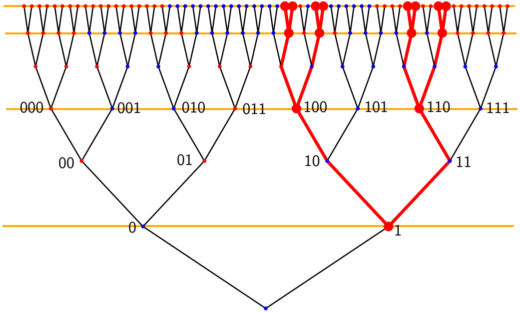

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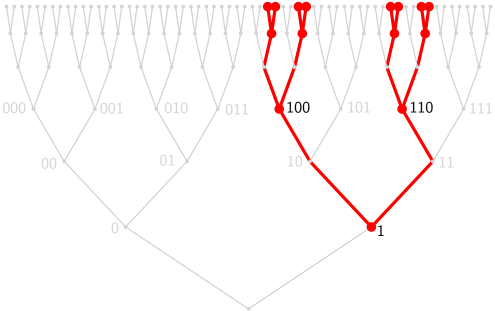

Notice that for regularly branching tree the strong subtree is isomorphic to the original tree.

Some more recent results on big Ramsey degrees

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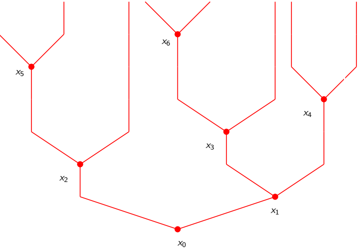
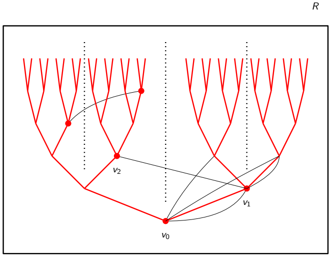
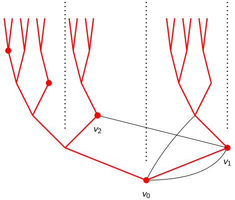
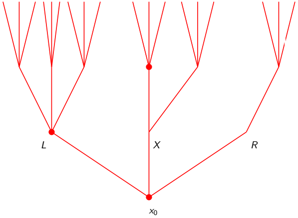
11. Bice, de Rancourt, H., Konečný: metric big Ramsey degrees of ℓ_∞ and the Urysohn sphere, (2023+).

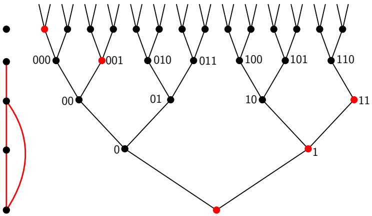
Big Ramsey degrees

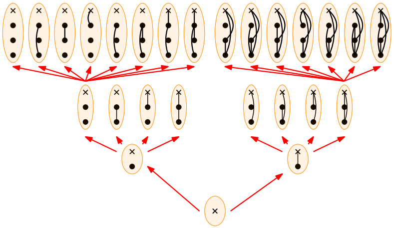












Big Ramsey degres by proof techniques

Ramsey's Theorem

ω, Unary languages

Ultrametric spaces

Λ-ultrametric

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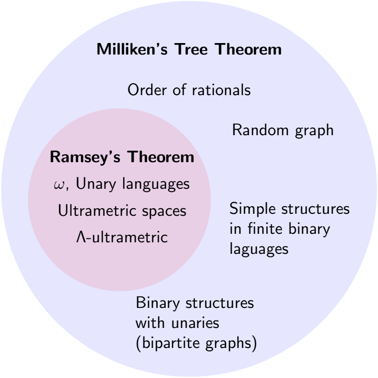

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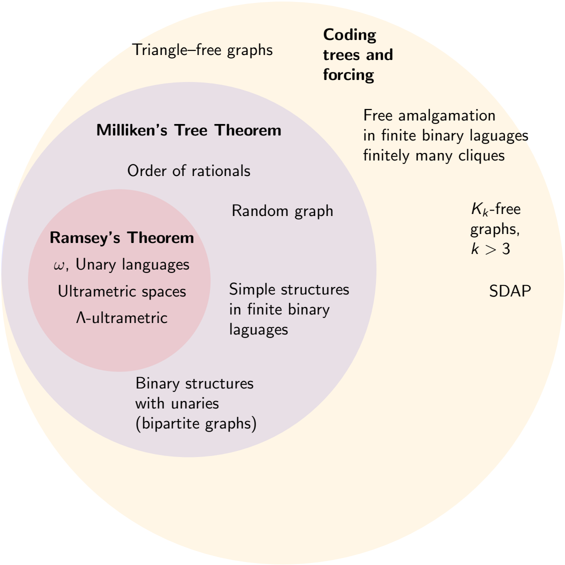

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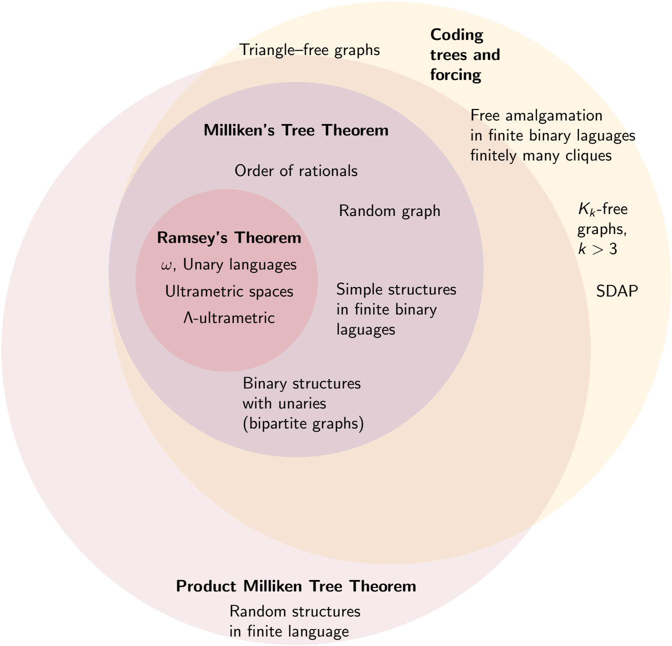

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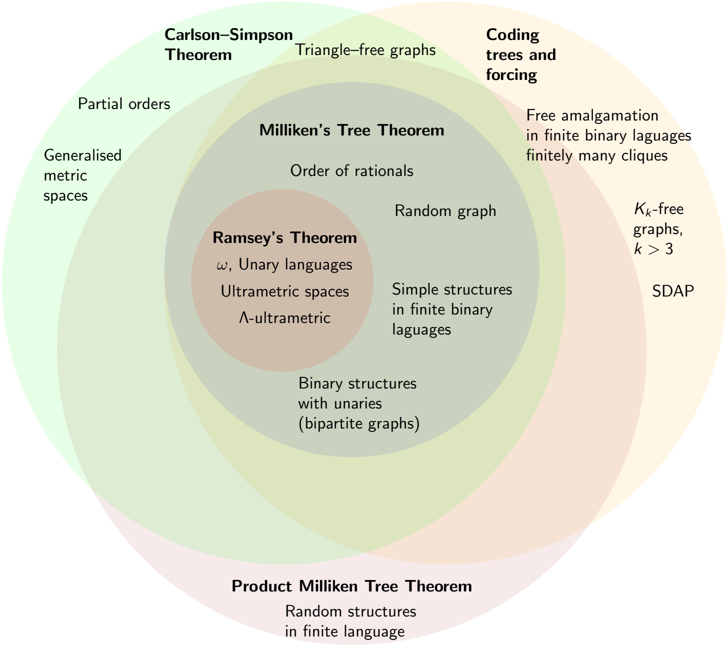

Trees with a successor operation

While most Ramsey-type theorems are concerned about regularly branching trees, we need more general notion allowing trees with finite but unbounded branching.

Definition ($\mathcal{S}$-tree)

An $\mathcal{S}$-tree is a quadruple $(T, \preceq, \Sigma, \mathcal{S})$ where $(T, \preceq)$ is a countable finitely branching tree with finitely many nodes of level 0, Σ is a set called the alphabet and $\mathcal{S}$ is a partial function $\mathcal{S}\colon T \times T^{<\omega} \times \Sigma \to T$ called the successor operation satisfying the following three axioms:

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Example: a binary tree

Consider $\mathcal{S}$-tree is $(2^{<\omega}, \sqsubseteq, \{0, 1\}, \mathcal{S})$.
$\mathcal{S}$ is defined only for empty parameters $\bar{p}$ by concatenation: $\mathcal{S}(a, c) = a^\frown c$.

$$\mathcal{S}(\mathcal{S}(\mathcal{S}(\mathcal{S}(\mathcal{S}((), 0), 1), 0), 1), 1) = 01011.$$

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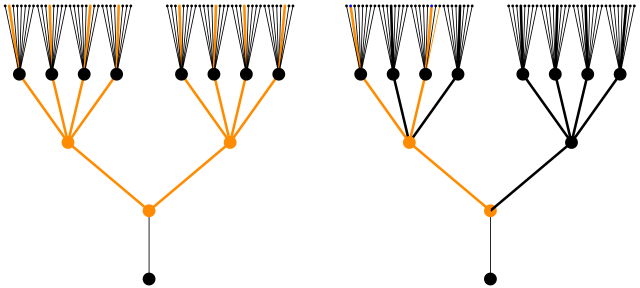

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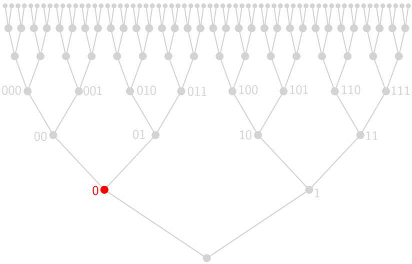

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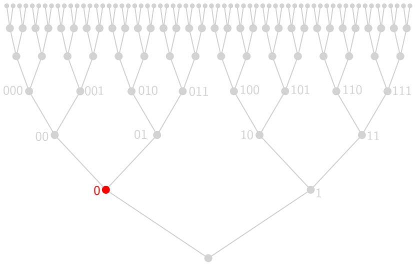

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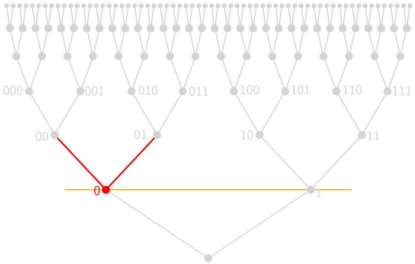

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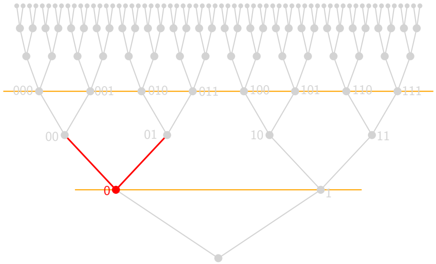

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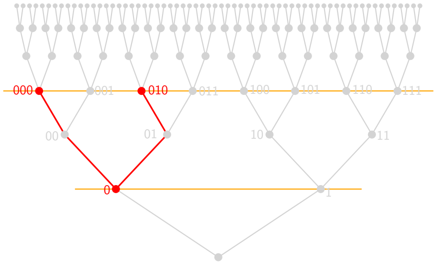

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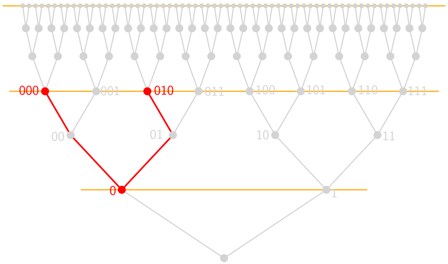
Given $S \subseteq T$, we also call a function $f\colon S \to T$ shape-preserving if it extends to a shape-preserving function $F\colon T \to T$.



Shape-preserving functions

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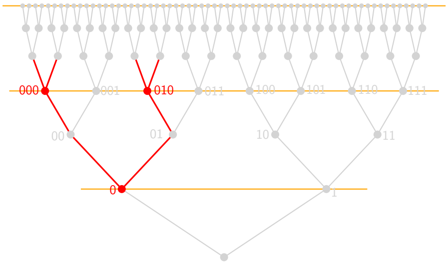

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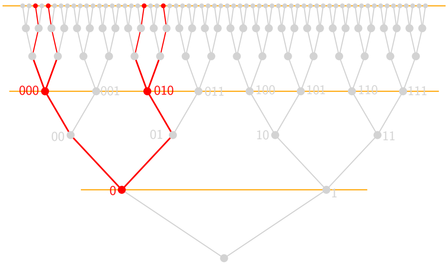

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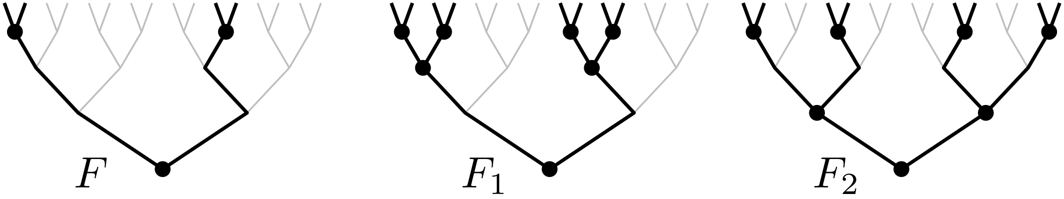

$\tilde{F}(0) = 0$, $\tilde{F}(1) = 2$: F skips levels 1 and 2.

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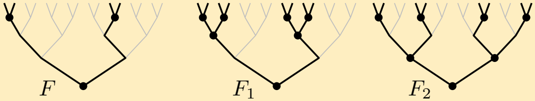

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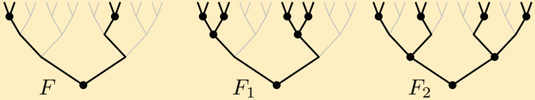

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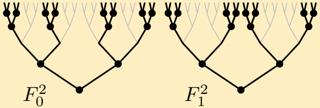

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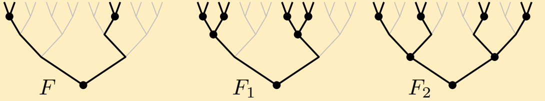

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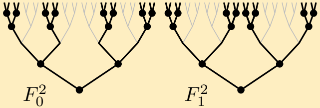

Ramsey theorem for trees with successor operation

Put $\mathcal{M}^n = \{F \in \mathcal{M} : F \restriction_{T(<n)} \text{ is identity}\}$, $\mathcal{AM}^n_k = \{F \restriction_{T(<n+k)} : F \in \mathcal{M}^n\}$.

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Ellentuck topology on $(\mathcal{S}, \mathcal{M})$-trees

Recall that a subset $\mathcal{X}$ of a topological space is

1. nowhere dense if every non-empty open set contains a non-empty open subset that avoids $\mathcal{X}$.
2. meager if is the union of countably many nowhere dense sets,
3. has the Baire property if it can be written as the symmetric difference of an open set and a meager set.

Put $\mathcal{AM} = \{F \restriction_{T(<n)} : F \in \mathcal{M}, n \in \omega\}$.

Definition (Ellentuck topological space $\mathcal{M}$)

Given an $(\mathcal{S}, \mathcal{M})$-tree $(T, \preceq, \Sigma, \mathcal{S}, \mathcal{M})$ we equip $\mathcal{M}$ with the Ellentuck topology given by the following basic open sets:

$$[f, F] = \{F \circ F' : F' \in \mathcal{M} \text{ and } F \circ F' \text{ extends } f\}$$

for every $f \in \mathcal{AM}$ and $F \in \mathcal{M}$.

Topological Ramsey theorem for trees with successor operation

Given a shape-preserving function $F \in \mathcal{M}$ and $f \colon T(\leq n) \to T$ such that $f \in \mathcal{AM}$ we define $\mathrm{depth}_F(f) = \tilde{g}(n)$ for $g \in \mathcal{AM}$ satisfying $F \circ g = f$. We set $\mathrm{depth}_F(f) = \omega$ if there is no such g.

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Let $\mathcal{X}$ be a subset of $\mathcal{M}$.

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Theorem (Ellentuck theorem for shape-preserving functions)

Let $(T, \preceq, \Sigma, \mathcal{S}, \mathcal{M})$ be an $(\mathcal{S}, \mathcal{M})$-tree and consider $\mathcal{M}$ with the Ellentuck topology. Then every property of Baire subset of $\mathcal{M}$ is Ramsey and every meager subset is Ramsey null.

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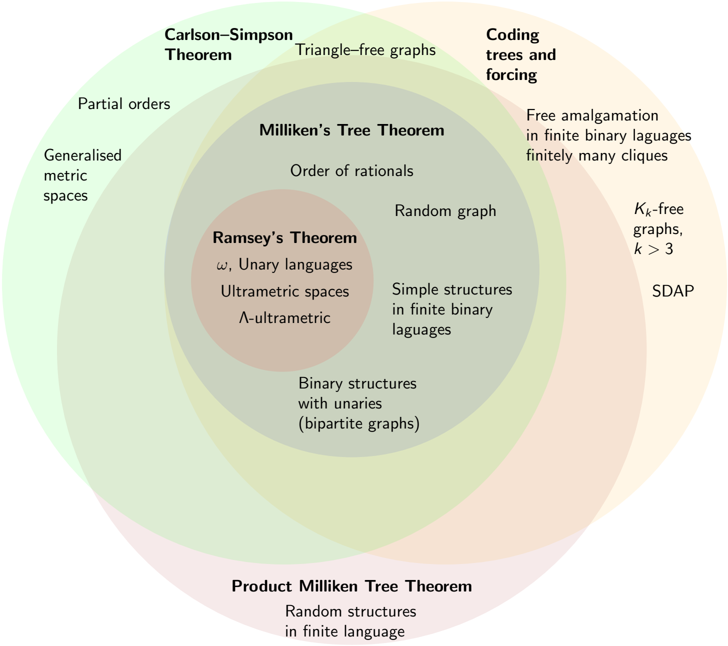
We obtain an interesting example of Ramsey space where Todorčević A3.2 axiom is not satisfied.

Applications to Big Ramsey degrees



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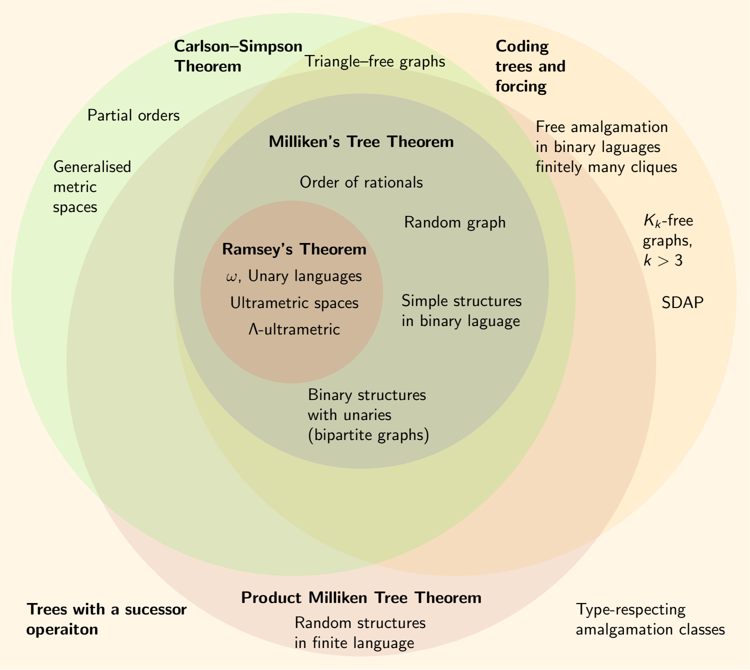
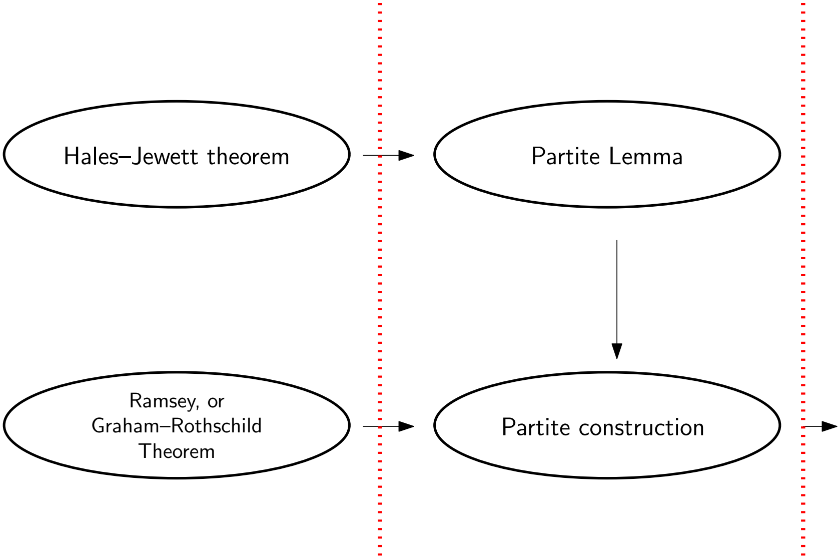


Applications to Big Ramsey degrees

Applications to small Ramsey degrees



1. Nešetřil–Rödl Theorem
2. Partial orders with linear extension
3. Ordered metric spaces
4. . . .

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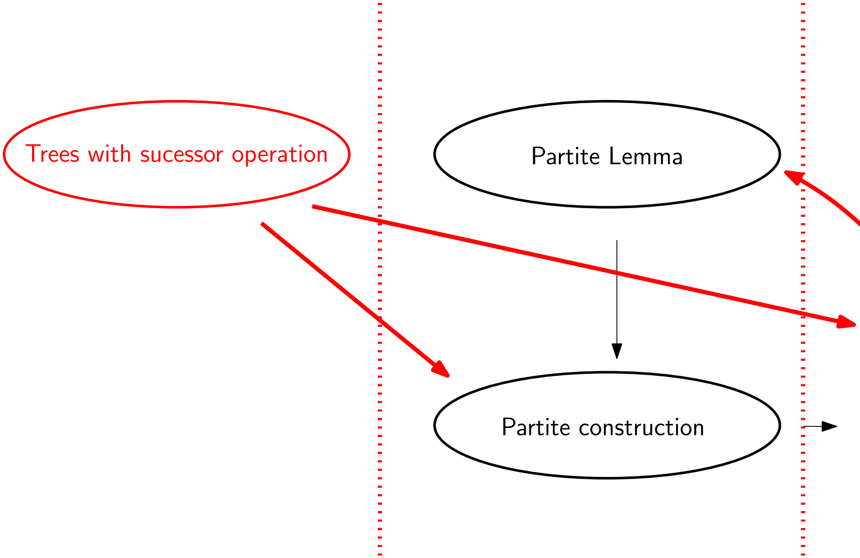

1. Abramson-Harrington (unrestricted Nešetřil–Rödl) theorem
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Abramson–Harrington theorem

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Let L be a relational language and $\mathbf{A}, \mathbf{B}$ finite ordered L-structures. Then there exists finite ordered L-structure $\mathbf{C}$ satisfying $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$.

Proof, step 1: associate vertices of structure B with words.

1. Fix **A** and **B**. WLOG assume that $B = n = |B|$ and $\leq_{\mathbf{B}}$ is the natural ordering of n.

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1. Fix $\mathbf{A}$ and $\mathbf{B}$. WLOG assume that $B = n = |B|$ and $\leq_{\mathbf{B}}$ is the natural ordering of n.
2. Given two substructures $\mathbf{B}'$ and $\mathbf{B}''$ of $\mathbf{B}$ we put $\mathbf{B}' \prec \mathbf{B}''$ if either $|\mathbf{B}'| < |\mathbf{B}''|$ or $|\mathbf{B}'| = |\mathbf{B}''|$ and B' is lexicographically before B'' (in the order of vertices of $\mathbf{B}$).

Abramson–Harrington theorem

Theorem (Nešetřil 1977, Abramson–Harrington 1978)

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3. Put $p = 2^n - 1$ and enumerate all non-empty substructures of $\mathbf{B}$ as $\mathbf{B}^0, \mathbf{B}^1, \ldots, \mathbf{B}^{p-1}$ in the increasing order (given by $\preceq$). For each $i < p$

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4. For each $i < N$ find lexicographically first substructure $\mathbf{D}^i$ isomorphic to $\mathbf{B}^i$ and denote by f^i the unique isomorphism $\mathbf{B}^i \to \mathbf{D}^i$.

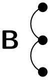
$$\varphi(v)_i = \begin{cases} -1 & \text{if } v \notin B^i \\ f^i(v) & \text{if } v \in B^i \end{cases} \quad \text{for every } v \in B \text{ and } i < p.$$

$\mathbf{B}$ 

		0	1	2	3	4	5	6
0	$\varphi(0) =$	0	n	n	0	0	n	0
1	$\varphi(1) =$	n	0	n	1	n	0	1
2	$\varphi(2) =$	n	n	0	n	2	1	2

□

Abramson–Harrington theorem



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Proof, step 2: structure $\mathbf{C}_\ell$ on Σ^ℓ.

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1. we say that $\bar{w}$ decides a structure on level $i < \ell$ if $0 \leq w^0_i < w^1_i < \cdots < w^{k-1}_i$ and i is a minimal with this property.

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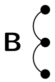

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   1. $k = 2$ and $w^0_{i'} \geq w^1_{i'} \geq 0$,
   2. $0 \leq w^0_{i'} < w^1_{i'} < \cdots < w^{k-1}_{i'}$ however there exists $i < i'$ such that $\bar{w}$ decides structure on level i and $B \restriction_{\{w^0_i, w^1_i, \dots, w^{k-1}_i\}}$ is not isomorphic to $B \restriction_{\{w^0_{i'}, w^1_{i'}, \dots, w^{k-1}_{i'}\}}$.

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B	0	$\varphi(0) =$	0	n	n	0	0	n	0
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For every $\ell \in \omega$ construct an ordered L-structure $\mathbf{C}_\ell$ as a structure satisfying the following:

1. The vertex set of $\mathbf{C}_\ell$ is $C_\ell = \Sigma^\ell$,
2. $\leq_{\mathbf{C}_\ell}$ is the lexicographic ordering of Σ^ℓ,
3. whenever $(w^0, w^1, \dots, w^{k-1}) \in \Sigma^\ell$ is compatible and decides structure on some level i then $B \restriction_{\{w^0, w^1, \dots, w^{k-1}\}}$ is isomorphic to $B \restriction_{\{w^0_i, w^1_i, \dots, w^{k-1}_i\}}$.

Abramson–Harrington theorem

Proof step 3: Building $(\mathcal{S}, \mathcal{M})$-tree.

Define successors by concatenation.
Let $\mathcal{M}$ denote the set of all shape-preserving functions $F\colon \Sigma^{<\omega} \to \Sigma^{<\omega}$ satisfying for every $\ell \in \omega$ and every lexicographically increasing sequence $\bar{w}$ of elements of Σ^{ℓ} the following two properties:

1. if $F(\bar{w})$ decides structure on level i then $i \in \tilde{F}[\omega]$.
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Let N by given by our theorem for $(\mathcal{S}, \mathcal{M})$-tree, $2^{|A|} - 1$ and $2^{|B|} - 1$. Then

$$\mathbf{C}_{\ell} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{2}.$$

□

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A

		0	1	2
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□

B (0, 1, 2)

	0	1	2	3	4	5	6
$\varphi(1) =$	~~n~~	0	n	~~1~~	~~n~~	0	~~1~~
$\varphi(2) =$	~~n~~	n	0	~~n~~	~~2~~	1	~~2~~

A (0, 1)

	0	1	2
$\varphi(0) =$	0	n	0
$\varphi(1) =$	n	0	1

Thank you for the attention

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