Combinatorial κ-Reals in the Higher Baire Space

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How do you show for two cardinal invariants $\mathfrak{x}$ and $\mathfrak{y}$ that $\mathfrak{x} < \mathfrak{y}$ is consistent?

Answer: (Usually) you assume $\mathfrak{x} = \mathfrak{y}$ are both ...

... **small** and add **witnesses** to increase $\mathfrak{y}$ without influencing $\mathfrak{x}$.
... **large** and add **witnesses** to decrease $\mathfrak{x}$ without influencing $\mathfrak{y}$.

For instance, we can force $\mathfrak{b} < \mathfrak{d}$ by ...

... adding $\aleph_2$-many **Cohen reals** over $\mathbf{V} \vDash$ "$\mathfrak{b} = \mathfrak{d} = \aleph_1$"
... adding $\aleph_1$-many **Cohen reals** over $\mathbf{V} \vDash$ "$\mathfrak{b} = \mathfrak{d} = \aleph_2$"

Question: Which forcing notions add which kinds of witnesses?

We will assume that κ is a regular uncountable cardinal. The **higher Baire space** ${}^\kappa\kappa$ is the set of κ-**reals** $f : \kappa \to \kappa$.

Many things known on ${}^\omega\omega$ also hold on ${}^\kappa\kappa$ by more or less the same proof, but not always! For example, $\mathfrak{b} < \mathfrak{s}$ is consistent, but $\mathfrak{s}_\kappa \leq \mathfrak{b}_\kappa$ is a theorem [Raghavan and Shelah, 2017].

Moreover, large cardinals are sometimes required to prove consistency. For example, $\mathfrak{s}_\kappa > \kappa^+$ implies the existence of a measurable μ with Mitchell order $\geq \mu^{++}$ [Zapletal, 1997].

Some notation:

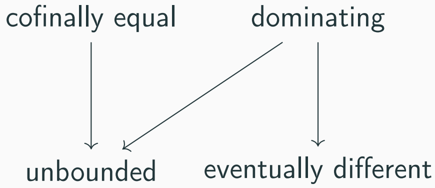
$$f \leq^* f' \quad \Leftrightarrow \quad \exists\alpha_0 \in \kappa \forall\alpha \geq \alpha_0 (f(\alpha) \leq f'(\alpha)),$$
$$f \leq^\infty f' \quad \Leftrightarrow \quad \forall\alpha_0 \in \kappa \exists\alpha \geq \alpha_0 (f(\alpha) \leq f'(\alpha)).$$

Similar for $=^*$, $=^\infty$, $\in^*$, $\in^\infty$.

Some κ-Reals

Let $\mathbf{V} \subseteq \mathbf{W}$ be models of ZFC. We call a κ-real $f \in ({}^\kappa\kappa)^\mathbf{W}$ ...

... **dominating** over $\mathbf{V}$ if $g \leq^* f$ for all $g \in ({}^\kappa\kappa)^\mathbf{V}$.
... **unbounded** over $\mathbf{V}$ if $f \not\leq^* g$ for all $g \in ({}^\kappa\kappa)^\mathbf{V}$.
... **eventually different** over $\mathbf{V}$ if $f \neq^\infty g$ for all $g \in ({}^\kappa\kappa)^\mathbf{V}$.
... **cofinally equal** over $\mathbf{V}$ if $f =^\infty g$ for all $g \in ({}^\kappa\kappa)^\mathbf{V}$.



An arrow $P \to Q$ means that "there exists a P κ-real over $\mathbf{V}$" implies "there exists a Q κ-real over $\mathbf{V}$".

κ-**Cohen forcing** $\mathbb{C}_\kappa$ has conditions $s \in {}^{<\kappa}\kappa$. The ordering is defined by $t \leq s$ iff $s \subseteq t$.

$\mathbb{C}_\kappa$ adds a κ-**Cohen generic** $\bigcup G \in {}^{\kappa}\kappa$, where G is a generic filter.

Theorem

A κ-Cohen generic is a cofinally equal κ-real.

Theorem

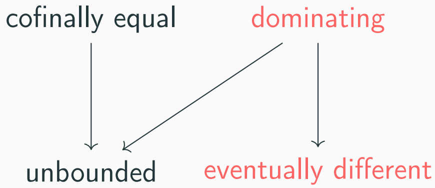
If $2^{<\kappa} = \kappa$, then $\mathbf{V}^{\mathbb{C}_\kappa}$ does not contain an eventually different κ-real over $\mathbf{V}$.

Proof sketch. Enumerate $\mathbb{C}_\kappa = \{p_\alpha \mid \alpha \in \kappa\}$ and given a name $\dot{f}$ for a κ-real, define a κ-real g such that $\Vdash$ "$\dot{f} =^\infty g$":

$$g : \alpha \mapsto \min\left\{\xi \,\middle|\, p_\alpha \nVdash \text{“}\dot{f}(\alpha) \neq \xi\text{”}\right\}$$

□

κ-Cohen Forcing



Let $\mathbf{V} \subseteq \mathbf{W}$ be models of ZFC with $b \in ({}^{\kappa}\kappa)^{\mathbf{V}}$. We assume $b(\alpha)$ is an infinite cardinal for all $\alpha \in \kappa$. Define:

$$\prod b = \prod_{\alpha \in \kappa} b(\alpha) = \{f \in {}^{\kappa}\kappa \mid f < b\}.$$

We call a bounded κ-real $f \in (\prod b)^{\mathbf{W}}$ ...

... b-**dominating** over $\mathbf{V}$ if $g \leq^* f$ for all $g \in (\prod b)^{\mathbf{V}}$.

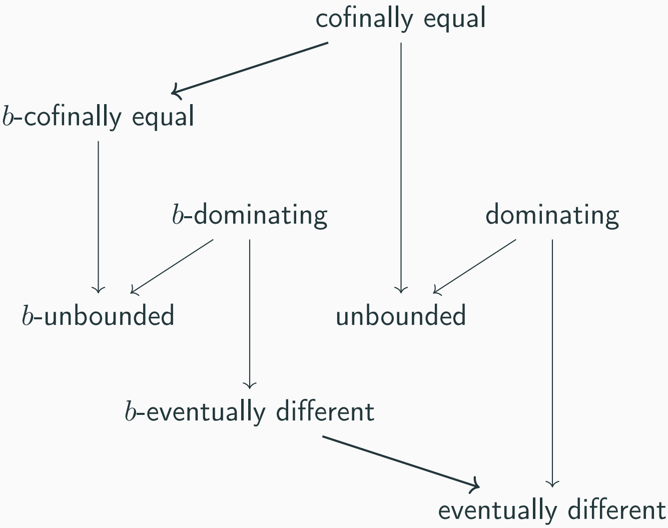
... b-**unbounded** over $\mathbf{V}$ if $f \not\leq^* g$ for all $g \in (\prod b)^{\mathbf{V}}$.

... b-**eventually different** over $\mathbf{V}$ if $f \neq^\infty g$ for all $g \in (\prod b)^{\mathbf{V}}$.

... b-**cofinally equal** over $\mathbf{V}$ if $f =^\infty g$ for all $g \in (\prod b)^{\mathbf{V}}$.

Note: b-dominating and b-unbounded κ-reals have no ${}^{\omega}\omega$-analogue.

Some Simple Observations



If $Q \subseteq \mathbb{P}$ is such that each $R \in [Q]^{<\kappa}$ has a lower bound (in $\mathbb{P}$), then Q is called $<\kappa$-**linked**. If $\mathbb{P}$ is the union of κ many $<\kappa$-linked sets, $\mathbb{P}$ is called $(\kappa, <\kappa)$-**centred**.

Lemma

If $\mathbb{P}$ is $(\kappa, <\kappa)$-centred, then $\mathbb{P}$ does not add b-eventually different κ-reals.

Proof sketch. Let $\mathbb{P}_\gamma$ be the $<\kappa$-linked subsets and $\Vdash$ " $\dot{f} \in \prod b$". We define f_γ s.t. if $h =^\infty f_\gamma$ for all $\gamma \in \kappa$, then $\Vdash$ " $h =^\infty \dot{f}$":

$$f_\gamma : \alpha \mapsto \min \left\{ \xi \;\middle|\; \forall p \in \mathbb{P}_\gamma (p \nVdash \text{" } \dot{f}(\alpha) \neq \xi \text{"}) \right\}. \qquad \square$$

κ-**Hechler forcing** $\mathbb{D}_\kappa$ has conditions (s, f) where $s \in {}^{<\kappa}\kappa$ and $f \in {}^{\kappa}\kappa$. The ordering is defined as $(t, g) \leq (s, f)$ iff $s \subseteq t$ and $f(\alpha) \leq g(\alpha)$ for all $\alpha \in \kappa \setminus \mathrm{dom}(s)$ and $f(\alpha) \leq t(\alpha)$ for all $\alpha \in \mathrm{dom}(t) \setminus \mathrm{dom}(s)$.

Theorem

$\mathbb{D}_\kappa$ adds a κ-Cohen generic and a dominating κ-real.

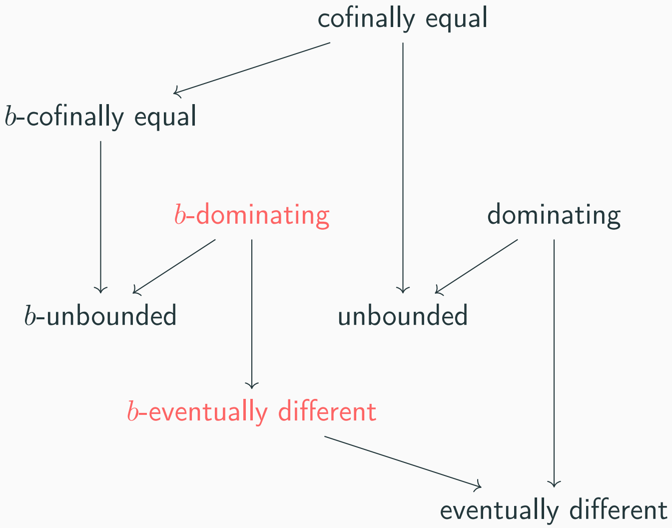
Theorem *[Cummings and Shelah, 1995, Lemma 7]*

If $2^{<\kappa} = \kappa$, then $\mathbb{D}_\kappa$ is $(\kappa, {<}\kappa)$-centred.

Corollary

If $2^{<\kappa} = \kappa$, then $\mathbb{D}_\kappa$ does not add b-eventually different κ-reals.

κ-Hechler Forcing



If $Q \subseteq \mathbb{P}$ is such that each $R \in [Q]^\kappa$ has some $R' \in [R]^\kappa$ such that R' has a lower bound (in $\mathbb{P}$), then Q is called κ-**calibre**. If $\mathbb{P}$ is the union of κ many κ-calibre sets, $\mathbb{P}$ is called (κ, κ)-**calibre**.

Lemma

If $\mathbb{P}$ is (κ, κ)-calibre, then $\mathbb{P}$ does not add dominating κ-reals.

Proof sketch. Let $\mathbb{P}_\gamma$ be the κ-calibre subsets and $\Vdash$ "$\dot{f} \in {}^\kappa\kappa$". We define f_γ s.t. if $f_\gamma \leq^\infty h$ for all $\gamma \in \kappa$, then $\Vdash$ "$\dot{f} \leq^\infty h$":

$$f_\gamma : \alpha \mapsto \min\left\{\xi \;\middle|\; \forall p \in \mathbb{P}_\gamma (p \nVdash \text{“}\dot{f}(\alpha) \geq \xi\text{”})\right\}$$

□

Bounded κ-Hechler Forcing

Let $b \in {}^\kappa\kappa$ be called **fast** if $\mathrm{cf}(b)$ is increasing and discontinuous on a club set C, i.e. $\mathrm{cf}(b(\alpha)) \leq \mathrm{cf}(b(\beta))$ for $\alpha \leq \beta$ and $\bigcup_{\xi\in\alpha} \mathrm{cf}(b(\xi)) < \mathrm{cf}(b(\alpha))$ for limit $\alpha \in C$.

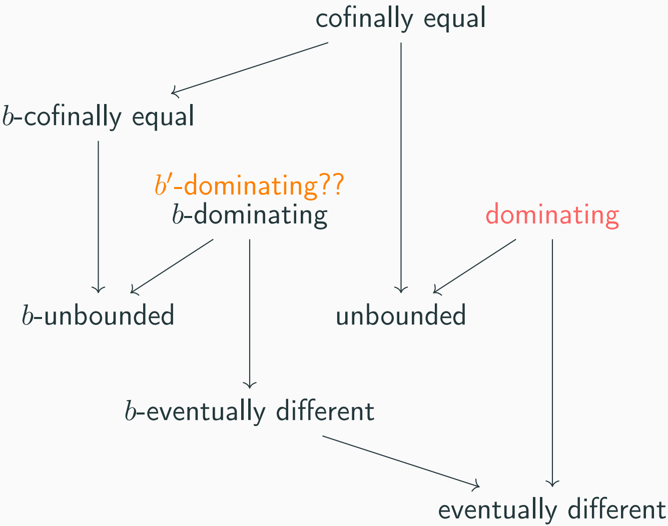
Bounded κ-Hechler forcing $\mathbb{D}^b_\kappa$ has conditions (s, f) where $s \in \prod_{<\kappa} b$ and $f \in \prod b$. The ordering is the same as in $\mathbb{D}_\kappa$.

Theorem

If b is fast, $\mathbb{D}^b_\kappa$ adds a κ-Cohen generic and a b-dominating κ-real.

Theorem *Follows from [Shelah, 2020]*

If κ is weakly compact and b is fast, $\mathbb{D}^b_\kappa$ has (κ, κ)-calibre, and hence does not add dominating κ-reals.


cofinally equal
b-cofinally equal
b'-dominating??
b-dominating
dominating
b-unbounded
unbounded
b-eventually different
eventually different

κ-**Eventually different forcing** $\mathbb{E}_\kappa$ has conditions (s, F) where $s \in {}^{<\kappa}\kappa$ and $F \in [{}^{\kappa}\kappa]^{<\kappa}$. The ordering is defined as $(t, G) \leq (s, F)$ iff $s \subseteq t$ and $F \subseteq G$ and $t(\alpha) \neq f(\alpha)$ for all $\alpha \in \mathrm{dom}(t) \setminus \mathrm{dom}(s)$ and $f \in F$.

Theorem

$\mathbb{E}_\kappa$ adds a κ-Cohen generic and an eventually different κ-real.

Lemma

If $2^{<\kappa} = \kappa$, then $\mathbb{E}_\kappa$ is $(\kappa, {<}\kappa)$-centred.

Theorem

$\mathbb{E}_\kappa$ does not add a b-eventually different κ-real.

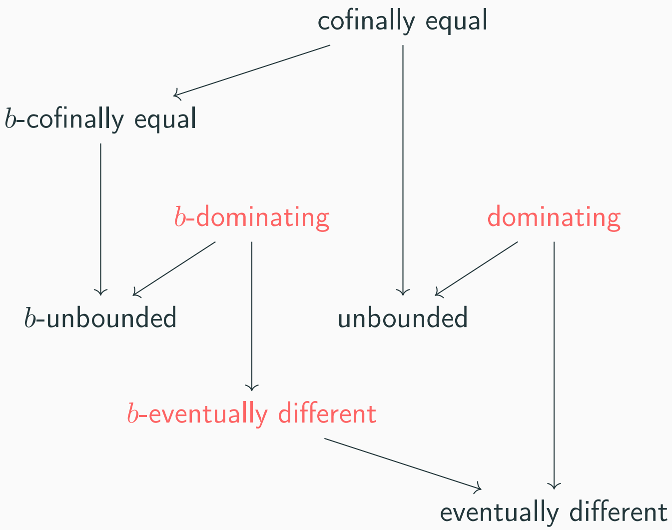
A space X is called **$<\kappa$-compact** if every cover of X has a subcover of size $<\kappa$. The **$<\kappa$-box topology** on $\prod_{i \in I} X_i$ is generated by open sets $[s] = \left\{ f \in \prod_{i \in I} X_i \mid s \subseteq f \right\}$ where $s \in \prod_{i \in J} X_i$ for $J \subseteq I$ with $|J| < \kappa$.

If κ is strongly compact, then the **κ-Tychonoff theorem** holds: the $<\kappa$-box product of $<\kappa$-compact spaces is $<\kappa$-compact.

Theorem *[Buhagiar and Džamonja, 2021, Theorem 5.1]*
The $<\kappa$-box product $\prod_{i \in \kappa} X_i$ with $w(X_i) \leq \kappa$ and X_i a $<\kappa$-compact space for each $i \in \kappa$ is $<\kappa$-compact if and only if κ is weakly compact.

Lemma *As in [Miller, 1981, Lemma 5.1]*
If κ is weakly compact, then $\mathbb{E}_\kappa$ does not add a dominating κ-real.

κ-Eventually Different Forcing



For $h \in {}^{\kappa}\kappa$, an h-**slalom** φ is an element of $\mathrm{Sl}^h_\kappa = \prod_{\alpha\in\kappa}[\kappa]^{<h(\alpha)}$. If $f \in {}^{\kappa}\kappa$, then $\varphi \in \mathrm{Sl}^h_\kappa$ **localises** f if $f \in^* \varphi$, i.e. $f(\alpha) \in \varphi(\alpha)$ for almost all $\alpha \in \kappa$.

A forcing notion $\mathbb{P}$ has the h-**Laver property** if for every $b \in {}^{\kappa}\kappa$, condition $p \in \mathbb{P}$ and name $\dot{f}$ with $p \Vdash$ "$\dot{f} \in \prod b$" there exists some $\varphi \in \mathrm{Sl}^h_\kappa$ and $q \le p$ such that $q \Vdash$ "$\dot{f} \in^* \varphi$".

Lemma

If $\mathbb{P}$ has the h-Laver property, then $\mathbf{V}^{\mathbb{P}}$ contains no κ-Cohen generics over $\mathbf{V}$ and no b-unbounded κ-reals for $\mathrm{cf}(b) > h$.

We say that $\mathbb{P}$ is ${}^{\kappa}\kappa$-**bounding**, if $\mathbf{V}^{\mathbb{P}}$ does not contain unbounded κ-reals over $\mathbf{V}$. We say $\mathbb{P}$ has the h-**Sacks property** if $\mathbb{P}$ is ${}^{\kappa}\kappa$-bounding and has the h-Laver property.

Forcing With Closed Perfect Trees

A tree $T \subseteq {}^{<\kappa}\kappa$ is …

… **perfect** if for all $u \in T$ there exists $v \in T$ with $u \subseteq v$ such that v is a splitting node.

… **closed** (under splitting) if for all chains $C \subseteq T$ of splitting nodes with $|C| < \kappa$, also $\bigcup C \in T$ is a splitting node.

… **guided by** $\mathcal{U} \subseteq \mathcal{P}(\kappa)$ if for every splitting node $u \in T$ the set $\{\alpha \in \kappa \mid u^\frown\alpha \in T\}$ is in $\mathcal{U}$.

… **Laver** if there is a stem $u \in T$ such that v is a splitting node iff $u \subseteq v$, for $v \in T$.

Let $\mathcal{U}$ be a nonprincipal $<\kappa$-complete normal filter on κ.

κ-**Laver forcing** $\mathbb{L}_\kappa^\mathcal{U}$ has the set of closed (perfect) Laver trees guided by $\mathcal{U}$ as conditions, ordered by $S \leq T$ if $S \subseteq T$.

Theorem

$\mathbb{L}_\kappa^\mathcal{U}$ adds a dominating κ-real and a κ-Cohen generic.

Theorem *[Khomskii, Koelbing, Laguzzi, and Wohofsky, 2022]*

Any subforcing $\mathbb{L} \subseteq \mathbb{L}_\kappa^\mathcal{U}$ closed under taking subtrees of the form $(T)_s$ for $s \in T \in \mathbb{L}$ adds a κ-Cohen generic.

Question

Does there exist a $<\kappa$-distributive forcing notion that adds a dominating κ-real, but no κ-Cohen generic?

Let $\mathcal{U}$ be a nonprincipal $<\kappa$-complete normal filter on κ.

κ-**Miller forcing** $\mathbb{M}\mathbb{i}_\kappa^{\mathcal{U}}$ has the set of closed perfect trees guided by $\mathcal{U}$ as conditions, ordered by $S \leq T$ if $S \subseteq T$.

Theorem

$\mathbb{M}\mathbb{i}_\kappa^{\mathcal{U}}$ adds an unbounded κ-real but no eventually different κ-real.

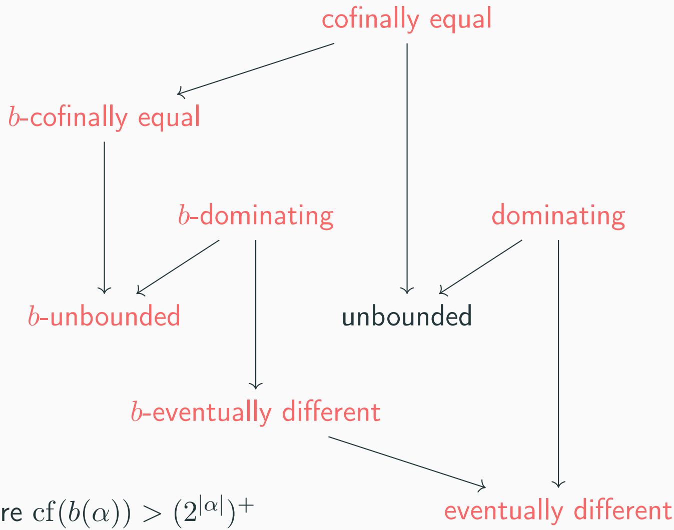
Theorem *[Brendle, Brooke-Taylor, Friedman, and Montoya, 2018, Prp. 77]*

If $\mathcal{U}$ is the club filter, then $\mathbb{M}\mathbb{i}_\kappa^{\mathcal{U}}$ adds a κ-Cohen generic.

Theorem *[Brendle, Brooke-Taylor, Friedman, and Montoya, 2018, Prp. 81]*

IIf $\mathcal{U}$ is an ultrafilter, then $\mathbb{M}\mathbb{i}_\kappa^{\mathcal{U}}$ has the h-Laver property for $h : \alpha \mapsto (2^{|\alpha|})^+$, hence does not add a κ-Cohen generic or b-unbounded κ-real for $\mathrm{cf}(b) > h$.

κ-Miller Forcing Guided by an Ultrafilter



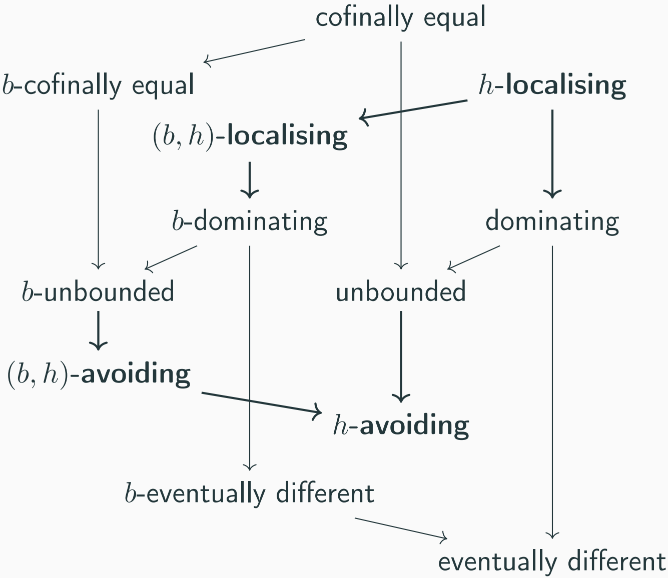
where $\mathrm{cf}(b(\alpha)) > (2^{|\alpha|})^+$

We call an h-slalom $\varphi \in (\mathrm{Sl}^h_\kappa)^{\mathbf{W}}$ h-**localising** over $\mathbf{V}$ if $f \in^* \varphi$ for all $f \in ({}^\kappa\kappa)^{\mathbf{V}}$.

We call a κ-real $f \in ({}^\kappa\kappa)^{\mathbf{W}}$ h-**avoiding** over $\mathbf{V}$ if $f \not\in^* \varphi$ for all $\varphi \in (\mathrm{Sl}^h_\kappa)^{\mathbf{V}}$.

By restricting ${}^\kappa\kappa$ to $\prod b$ we can also define (b, h)-**localising** slaloms and (b, h)-**avoiding** κ-reals.

Some More Simple Observations



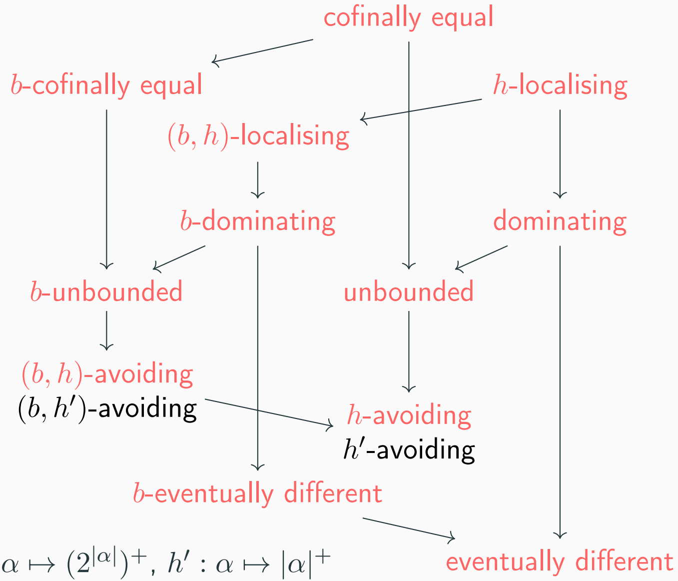
κ-**Sacks forcing** $\mathbb{S}_\kappa$ has the set of closed perfect trees as conditions, ordered by $S \leq T$ if $S \subseteq T$.

Theorem

$\mathbb{S}_\kappa$ does not add eventually different κ-reals.

Theorem *[Brendle, Brooke-Taylor, Friedman, and Montoya, 2018, Lm. 69]*

$\mathbb{S}_\kappa$ has the h-Sacks property for $h : \alpha \mapsto (2^{|\alpha|})^+$ but adds an h'-avoiding κ-real for $h' : \alpha \mapsto |\alpha|^+$.



where $h : \alpha \mapsto (2^{|\alpha|})^+$, $h' : \alpha \mapsto |\alpha|^+$

κ-Miller Lite Forcing

Let $h \in {}^\kappa\kappa$ be a cofinally increasing function with cardinal values.

κ-**Miller Lite forcing** $\mathbb{ML}_\kappa^h$ has the set of closed perfect trees T as conditions such that splitting nodes $u \in T$ with (order-type) α many splitting nodes below u split into $h(\alpha)$-many successors.

The ordering is given by $S \leq T$ if $S \subseteq T$ and $\mathrm{suc}(u, S) \neq \mathrm{suc}(u, T)$ implies $|\mathrm{suc}(u, S)| < |\mathrm{suc}(u, T)|$ for all splitting nodes $u \in S$.

Theorem

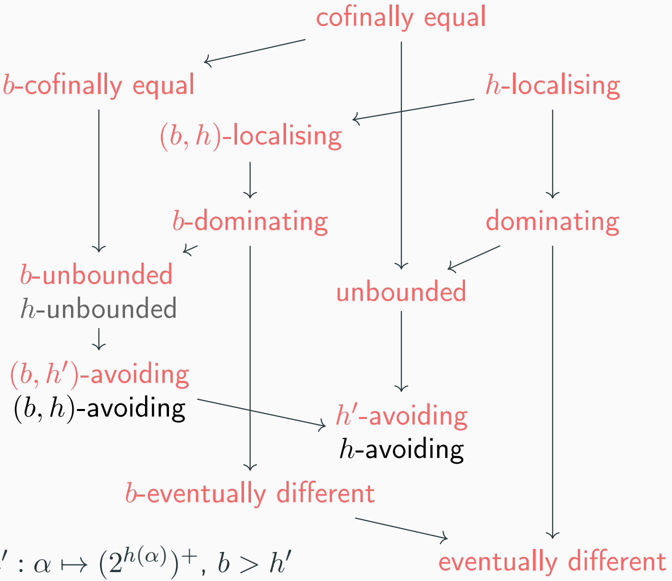
$\mathbb{ML}_\kappa^h$ does not add eventually different κ-reals.

Theorem *[vdV.]*

$\mathbb{ML}_\kappa^h$ has the $(2^h)^+$-Sacks property, but adds an h-avoiding κ-real.

Theorem

$\mathbb{ML}_\kappa^h$ adds an h-unbounded κ-real.



where $h' : \alpha \mapsto (2^{h(\alpha)})^+$, $b > h'$

There are many more things to check. Most are likely easy, some could be hard. Some examples:

Can you add b-dominating κ-reals without adding b'-dominating κ-reals?

Are eventually different κ-reals in $\mathbf{V}^{\mathbb{D}_\kappa}$ dominating? (on ${}^\omega\omega$: yes)

Does κ-Miller forcing add a κ-Sacks generic? (on ${}^\omega\omega$: no)

If d is a dominating κ-real over $\mathbf{V}$ and c is κ-Cohen generic over $\mathbf{V}[d]$, is $d + c$ then $\mathbb{D}_\kappa$-generic over $\mathbf{V}$? (on ${}^\omega\omega$: yes)

What about splitting κ-reals?

What about random κ-reals?

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