Orderings on P-point ultrafilters

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Joint work with Dilip Raghavan and Jonathan Verner

Definition (Gillman-Henriksen, 1954)

For a topological space X, we say that $x \in X$ is a *P-point* if the only prime ideal of $C(X, \mathbb{R})$ at x is the maximal ideal M_x.

Note that this is equivalent to the fact that intersection of any countably many neighborhoods of x is a neighborhood of x.

Again, in the space $\beta\omega$, this is equivalent to saying that a non-principal ultrafilter $\mathcal{U}$ is a P-point iff for any collection $\{a_n : n < \omega\} \subseteq \mathcal{U}$ there is an $a \in \mathcal{U}$ such that $a \setminus a_n$ is finite for all $n < \omega$.

Typically, one would write $a \subseteq^* a_n$ instead of saying that $a \setminus a_n$ is finite.

Another equivalent condition for a non-principal ultrafilter $\mathcal{U}$ to be a P-point is that for every function $f : \omega \to \omega$ either f is finite-to-one on an element of $\mathcal{U}$ or f is constant on an element of $\mathcal{U}$.

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Theorem (W. Rudin, 1956)

Assume CH*. Then there is a P-point ultrafilter in* $\beta\omega$*. Thus* ω^* *is not a homogeneous space.*

Theorem (Shelah, 1970s)

There is a model of ZFC *with no P-point ultrafilters.*

$\mathrm{MA}(\sigma$-centered) ensures the existence of $2^{\mathfrak{c}}$ many P-points. Note that

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Definition

Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters on ω. We say that $\mathcal{U}$ is *Rudin-Keisler reducible* to $\mathcal{V}$ if there is a function $f : \omega \to \omega$ such that for every $a \subseteq \omega$:

$$a \in \mathcal{U} \Leftrightarrow f^{-1}[a] \in \mathcal{V}.$$

We also say that $\mathcal{U}$ is Rudin-Keisler below $\mathcal{V}$ and write $\mathcal{U} \leq_{RK} \mathcal{V}$.

Note that this is equivalent to the condition that $f[a] \in \mathcal{U}$ for each $a \in \mathcal{V}$. This motivates the notation $f(\mathcal{V}) = \mathcal{U}$ which is sometimes used.

Observation

For ultrafilters $\mathcal{U}$ and $\mathcal{V}$ there is a bijection $f : \omega \to \omega$ such that $a \in \mathcal{U} \Leftrightarrow f^{-1}[a] \in \mathcal{V}$ if and only if $\mathcal{U} \leq_{RK} \mathcal{V}$ and $\mathcal{V} \leq_{RK} \mathcal{U}$.

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Theorem (Kunen, 1970)

There are ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on ω such that $\mathcal{U} \not\leq_{RK} \mathcal{V}$ and $\mathcal{V} \not\leq_{RK} \mathcal{U}$.

Note that if $\mathcal{V}$ is a P-point and $\mathcal{U} \leq_{RK} \mathcal{V}$, then $\mathcal{U}$ is also a P-point.

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Definition

An ultrafilter $\mathcal{U}$ is *selective* (Ramsey) iff for every function $f : \omega \to \omega$, either f is 1-1 on an element of $\mathcal{U}$ or f is constant on an element of $\mathcal{U}$.

Theorem (Blass, 1970)

An ultrafilter $\mathcal{U}$ on ω is selective iff it is minimal in the RK ordering. (i.e. $\mathcal{U}$ is selective iff for any non-principal $\mathcal{V}$: $\mathcal{V} \leq_{RK} \mathcal{U} \Rightarrow \mathcal{U} \leq_{RK} \mathcal{V}$).

Note that Kunen showed in the early 1970s that there are no selective ultrafilters in the random real model.

Theorem (Keisler, early 1970s)

Under CH, there are $2^{\mathfrak{c}}$ pairwise RK incomparable selective ultrafilters.

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Question (Raghavan-Shelah, 2017)

Assume $\mathrm{MA}(\sigma - \text{centered})$. Let $\mathbb{P}$ be a partial order of size at most $2^{\mathfrak{c}}$ where every element has at most $\mathfrak{c}$ many predecessors. Does $\mathbb{P}$ embed into the set of P-points under the *RK* ordering (and under the Tukey ordering).

Theorem (Blass, 1973)

Both ω_1 and $(\mathbb{R}, <)$ embed into the set of P-points under the RK ordering.

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If a countable set $\{\mathcal{U}_n : n < \omega\}$ of P-points has an upper bound which is a P-point, then there is an ultrafilter $\mathcal{U}$ such that $\mathcal{U} \leq_{RK} \mathcal{U}_n$ for each $n < \omega$.

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Theorem (Rosen, 1985)

Ordinal ω_1 embeds into the set of P-points under the RK ordering as an initial segment, i.e. there is a set of P-points $\{\mathcal{U}_\alpha : \alpha < \omega_1\}$ such that

- $\mathcal{U}_\alpha <_{RK} \mathcal{U}_\beta$ *for all* $\alpha < \beta < \omega_1$ *and*
- *for any ultrafilter* $\mathcal{U}$*, if there is* $\alpha < \omega_1$ *such that* $\mathcal{U} \leq_{RK} \mathcal{U}_\alpha$*, then there is some* $\gamma < \omega_1$ *such that* $\mathcal{U}_\gamma \equiv_{RK} \mathcal{U}$.

Theorem (Laflamme, 1989)

For each $1 \leq \alpha < \omega_1$, there is an ultrafilter $\mathcal{U}_\alpha$, generic for a partial order $\mathbb{P}_\alpha$ with the following properties:

- $\mathcal{U}_\alpha$ *is a rapid P-point ultrafilter.*
- *There is a sequence* $\langle \mathcal{V}_\gamma : \gamma < \alpha + 1 \rangle$ *of P-points such that* $\mathcal{V}_0 = \mathcal{U}_\alpha$*, that* $\mathcal{V}_\gamma <_{RK} \mathcal{V}_\beta$ *for all* $\beta < \gamma < \alpha + 1$*, and that for any* $\mathcal{U}$ *with* $\mathcal{U} \leq_{RK} \mathcal{U}_\alpha$ *there is* $\gamma < \alpha + 1$ *such that* $\mathcal{U} \equiv_{RK} \mathcal{V}_\gamma$.

Theorem (Raghavan-Shelah, 2017)

Assume $\mathrm{MA}(\sigma - \textit{centered})$*. Then* $(P(\omega)/Fin, \subseteq^*)$ *embeds into the set of P-points under the RK ordering (and under the Tukey ordering as well).*

In particular, this implies that every poset of size at most $\mathfrak{c}$ embeds.

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Assume CH*. Then* $\mathfrak{c}^+$ *embeds into the set of P-points under the RK ordering (and under the Tukey ordering as well).*

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Theorem (Raghavan-Verner, 2019)

Assume CH. *Let* $\delta < \omega_2$ *and let* $\langle \mathcal{U}_\gamma : \gamma < \delta \rangle$ *be an RK increasing sequence of rapid P-points. Then there is a rapid P-point such that* $\mathcal{U}_\gamma \leq_{RK} \mathcal{U}$ *for every* $\gamma < \delta$.

(Some assumption on $\mathcal{U}_\gamma$*'s is needed, as they also prove that there is an* ω_1*-sequence of P-points which cannot be further extended.)*

Theorem (Starosolski 2021)

Assume $\mathfrak{b} = \mathfrak{c}$. *Then:*

1. *If* $\langle \mathcal{U}_n : n < \omega \rangle$ *is an RK-increasing sequence of P-point ultrafilters, then there is a P-point* $\mathcal{U}$ *such that* $\mathcal{U}_n <_{RK} \mathcal{U}$ *for each* $n < \omega$.
2. *For each P-point* $\mathcal{U}$, *there is an embedding of both the real line and the long line in the RK-ordering of P-points above* $\mathcal{U}$.
3. *For every P-point* $\mathcal{U}$ *and every* $\gamma < \mathfrak{c}^+$ *there is an RK-increasing sequence of P-points* $\langle \mathcal{U}_\alpha : \alpha < \gamma \rangle$ *such that* $\mathcal{U}_0 = \mathcal{U}$.

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The language L consists of symbols for all relations and all functions on ω.

$\mathcal{N}$ is the standard model for this language.

$\mathcal{M}$ is an elementary extension of $\mathcal{N}$.

Note that if $a \in \mathcal{M}$, then the set $\{{}^*f(a) : f \in \omega^\omega\}$ is the domain of an elementary submodel of $\mathcal{M}$. Submodels like this are called *principal*.

Then a principal submodel generated by a is isomorphic to the ultrapower of the standard model by the ultrafilter $\mathcal{U}_a = \{b \subseteq \omega : a \in {}^*b\}$, i.e.

$$\mathcal{N}^\omega / \mathcal{U}_a \cong \{{}^*f(a) : f \in \omega^\omega\}.$$

For $A, B \subseteq \mathcal{M}$, we say that they are *cofinal with each other* iff

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There is a reformulation of being a P-point in model theoretic terms.

Lemma

An ultrafilter $\mathcal{U}$ on ω is a P-point if and only if every nonstandard elementary submodel of $\mathcal{N}^\omega/\mathcal{U}$ is cofinal with $\mathcal{N}^\omega/\mathcal{U}$.

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Theorem (Blass, 1973)

If a countable set $\{\mathcal{U}_n : n \in \omega\}$ of P-points has an upper bound which is a P-point, then there is an ultrafilter $\mathcal{U}$ such that $\mathcal{U} \leq_{RK} \mathcal{U}_n$ for each $n \in \omega$.

This theorem has two immediate consequences.

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Any RK-decreasing ω-sequence of P-points has an RK-lower bound.

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If two P-points have an upper bound, then they also have a lower bound.

The latter may be viewed as a witness to the fact that, under MA for example, the *RK* ordering of P-points is not upwards directed.

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An ultrafilter $\mathcal{U}$ on ω is a $P_{\mathfrak{c}}$-point if for any $\alpha < \mathfrak{c}$ and any collection $\{a_i : i < \alpha\} \subseteq \mathcal{U}$ there is some $a \in \mathcal{U}$ such that $a \subseteq^* a_i$ for each $i < \alpha$.

Lemma

Let $\alpha < \mathfrak{c}$ and $\{\mathcal{M}_i : i < \alpha\}$ be a collection of submodels of $\mathcal{M}$ such that:

- *each $\mathcal{M}_i$ is generated by a_i,*
- *$\mathcal{M}_j \subseteq \mathcal{M}_i$ whenever $i < j < \alpha$,*
- *each $\mathcal{M}_i$ is cofinal with $\mathcal{M}_0$.*

Suppose moreover that $\mathcal{U}_0 = \{b \subseteq \omega : a_0 \in {}^*b\}$ is a $P_{\mathfrak{c}}$-point. Then there is a family $\{f_i : i < \alpha\} \subseteq \omega^\omega$ of finite-to-one maps such that:

- *${}^*f_i(a_0) = a_i$ for $i < \alpha$,*
- *for $i < j < \alpha$, there is $k \in \omega$ such that $f_j(n) = f_j(m)$ whenever $f_i(n) = f_i(m)$ and $m, n \geq k$.*

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Let α be an ordinal, let $\mathcal{F} = \{f_i : i < \alpha\} \subseteq \omega^\omega$ be a family of functions, and let A be a subset of α. We say that a set $d \subseteq \omega$ is $(A, \mathcal{F})$-*closed* if $f_i^{-1}[f_i[d]] \subseteq d$ for each $i \in A$.

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Let $\alpha < \mathfrak{c}$ and let $\mathcal{F} = \{f_i : i < \alpha\} \subseteq \omega^\omega$ be a family of finite-to-one maps. Suppose that for each $i < j < \alpha$ there is $k \in \omega$ such that $f_j(n) = f_j(m)$ whenever $f_i(n) = f_i(m)$ and $n, m \geq k$. Then for each finite $A \subseteq \alpha$ and each $w \in \omega$, there is a finite $(A, \mathcal{F})$-closed set $d \subseteq \omega$ such that $w \in d$.

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Assume MA_α. *Let* $\mathcal{F} = \{f_i : i < \alpha\} \subseteq \omega^\omega$ *be a family of finite-to-one functions. Suppose that for each non-empty finite set* $A \subseteq \alpha$, *and each* $w \in \omega$, *there is a finite* $(A, \mathcal{F})$*-closed set* $d \subseteq \omega$ *containing* w *as an element. Then there is a finite-to-one function* $h \in \omega^\omega$, *and a collection* $\{e_i : i < \alpha\} \subseteq \omega^\omega$ *such that for each* $i < \alpha$, *there is* $k \in \omega$ *such that* $h(n) = e_i(f_i(n))$ *whenever* $n \geq k$.

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Assume MA_α. *Let* $\mathcal{M}_i$ $(i < \alpha)$ *be a collection of pairwise cofinal submodels of* $\mathcal{M}$. *Suppose that* $\mathcal{M}_0$ *is principal, and that* $\mathcal{U}_0 = \{b \subseteq \omega : a_0 \in {}^*b\}$ *is a* $P_\mathfrak{c}$*-point, where* a_0 *generates* $\mathcal{M}_0$. *Then there is an element* $c \in \bigcap_{i<\alpha} \mathcal{M}_i$ *which generates a principal model cofinal with all* $\mathcal{M}_i$.

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Assume MA. If a collection of fewer than $\mathfrak{c}$ many $P_\mathfrak{c}$-points has an upper bound which is a $P_\mathfrak{c}$-point, then it has a lower bound.

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https://sites.dmi.uns.ac.rs/settop/2024/