Antichain numbers of $\mathcal{P}(\omega)/\mathcal{J}$

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$(+)$-covering number

Suppose that $\mathcal{J}$ is a tall, homogeneous ideal on ω. Recall

$$add^*(\mathcal{J}) = \text{the minimal size of a family } \mathcal{F} \subseteq \mathcal{J} \text{ with no pseudo-union in } \mathcal{J},$$

$$cov^*(\mathcal{J}) = \text{the minimal size of a family } \mathcal{F} \subseteq \mathcal{J} \text{ such that every infinite set infinitely intersects some element of } \mathcal{F}.$$

(+)-covering number

We define then the (+)-covering number of $\mathcal{J}$ as

$cov^*_+(\mathcal{J})$ = the minimal size of a family $\mathcal{F} \subseteq \mathcal{J}$ such that every $\mathcal{J}$-positive set infinitely intersects some element of $\mathcal{F}$.

Observe that $add^*(\mathcal{J}) \leq cov^*_+(\mathcal{J}) \leq cov^*(\mathcal{J})$

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- B. Farkas, L. Zdomskyy *"Ways of destruction"* (2022).
- B. Balcar, F. Hernández-Hernández, M. Hrušák; *"Combinatorics of dense subsets of the rationals"* (2004)
- A. Marton; *"P-like properties of meager ideals and cardinal invariants"* (2004)

Antichain number

Let $\mathbb{B}$ be a Boolean algebra. The antichain number of $\mathbb{B}$ is defined as

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We will write $\mathfrak{a}(J)$ for $\mathfrak{a}(\mathcal{P}(\omega)/J)$

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- (B.Farkas, L.Soukup 2010) $\mathfrak{b} \leq \mathfrak{a}(J)$ for analytic P-ideals

Antichain numbers vs $cov^*_+(\mathcal{J})$

Theorem

Suppose that an ideal $\mathcal{J}$ is good. Then we have

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$\mathcal{J}$	$add^*(\mathcal{J})$	$cov^*_+(\mathcal{J})$	$cov^*(\mathcal{J})$
nwd	ω	$add(\mathcal{M})$	$cov(\mathcal{M})$
$fin \times fin$	ω	ω	$\mathfrak{b}$
$\mathcal{ED}_{fin}$	ω	$min\{\mathfrak{b}, cov^*(\mathcal{ED}_{fin})\} \leq$	$non(\mathcal{M})$
$\mathcal{Z}$	$add(\mathcal{N})$	?	$\leq non(\mathcal{M})$
$\mathcal{S}ol$	ω	$non(\mathcal{N})$	$non(\mathcal{N})$
$\mathcal{R}$	ω	$\mathfrak{c}$	$\mathfrak{c}$
$conv$	ω	$\mathfrak{c}$	$\mathfrak{c}$

Proof

Theorem

Suppose that $\mathcal{J}$ is a good ideal. Then $\mathfrak{a}(\mathcal{J}) \geq min\{\mathfrak{b}, cov_+^*(\mathcal{J})\}$.

Let $\kappa < min\{\mathfrak{b}, cov_+^*(\mathcal{J})\}$.
Let $\{A_\alpha : \alpha < \kappa\} \subseteq \mathcal{J}^+$ be such that $A_\alpha \cap A_\beta \in \mathcal{J}$ for $\alpha \neq \beta$.
We will construct $C \in \mathcal{J}^+$ such that $C \cap A_\alpha \in \mathcal{J}$ for all α's.

1. Using $\kappa < cov^*_+(\mathcal{J})$ find $\mathcal{J}$-positive B_α's such that $B_\alpha \subseteq A_\alpha$ and $B_\alpha \cap A_\beta$ is finite for $\alpha \neq \beta$.

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2. (goodness of $\mathcal{J}$) Every unctbl almost disjoint family $\mathcal{B} \subseteq \mathcal{J}^+$ has a ctbl subfamily $\{B_n : n \in \omega\} \subseteq \mathcal{B}$ such that:
for every $f \in \omega^\omega$ there is a sequence $\{C_n : n \in \omega\} \subseteq \mathcal{J}$, $C_n \subseteq B_n \setminus f(n)$ such that $C := \bigcup_n C_n$ is $\mathcal{J}$-positive.

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3. Let $f_\alpha \in \omega^\omega$ be such that $B_n \cap A_\alpha \subseteq f_\alpha(n)$. Using $\kappa < \mathfrak{b}$ find $f \in \omega^\omega$ dominating all f_α's. Use goodness to find desired C.

Which ideals are good?

- *nwd*
- $\varnothing \times fin$ and $fin \times fin$
- all F_σ ideals, in particular: $\mathcal{ED}$, Random graph and Solecki ideal, Van Der Waerden ideal
- all analytic P-ideals:
- conv, Ramsey ideal

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Which Borel (analytic, coanalytic) ideal is not good?

Thank you for your attention, but ...

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Thank you
Let's go eat