Big Ramsey degrees, structures, dynamics III

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January/February 2024
Winter School
Hejnice, Czech Republic

Outline of part III.

- Big Ramsey degrees: history and examples.

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- Completion flows.

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- Big Ramsey structures.
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- A dynamical object without BRS? When BRD is infinite?

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Definition

Given $\mathcal{L}$-structures $\mathbf{A} \leq \mathbf{B} \leq \mathbf{C}$ and $0 < k, r < \omega$, we write $\mathbf{C} \to (\mathbf{B})^{\mathbf{A}}_{r,k}$ if whenever $\gamma \colon \mathrm{Emb}(\mathbf{A}, \mathbf{C}) \to r$ is a coloring, there is $x \in \mathrm{Emb}(\mathbf{B}, \mathbf{C})$ with $|\{\gamma(x \circ f) : f \in \mathrm{Emb}(\mathbf{A}, \mathbf{B})\}| \leq k$.

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For instance, given $\mathbf{A} \in \mathcal{K}$, we have $\mathrm{SRD}(\mathbf{A}, \mathcal{K}) = t < \omega$ iff for every $\mathbf{A} \leq \mathbf{B} \in \mathcal{K}$ and $0 < r < \omega$, there is $\mathbf{B} \leq \mathbf{C} \in \mathcal{K}$ such that $\mathbf{C} \to (\mathbf{B})^{\mathbf{A}}_{r,t}$.

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Equivalently, this happens iff for every such $\mathbf{B}$ and r, we have $\mathbf{K} \to (\mathbf{B})^{\mathbf{A}}_{r,t}$.

Definition

Fix a countably infinite $\mathcal{L}$-structure $\mathbf{M}$ and some finite $\mathbf{A} \leq \mathbf{M}$. The big Ramsey degree of $\mathbf{A}$ in $\mathbf{M}$, denoted $\mathrm{BRD}(\mathbf{A}, \mathbf{M})$, is the least $t < \omega$ (if it exists) so that for every $0 < r < \omega$, we have $\mathbf{M} \rightarrow (\mathbf{M})^{\mathbf{A}}_{r,t}$. Otherwise put $\mathrm{BRD}(\mathbf{A}, \mathbf{M}) = \infty$.

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If $\mathcal{K}$ is a Fraïssé class with limit $\mathbf{K}$ and $\mathbf{A} \in \mathcal{K}$, we can write $\mathrm{BRD}(\mathbf{A}, \mathcal{K})$ for $\mathrm{BRD}(\mathbf{A}, \mathbf{K})$. Say $\mathcal{K}$ has finite BRD if $\mathrm{BRD}(\mathbf{A}, \mathcal{K})$ is always finite.

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$\mathbf{M}$ need not be ultrahomogeneous! If $\mathbf{M} = \langle \omega, < \rangle$, the ordinary infinite Ramsey theorem can be phrased as saying that for any finite linear order $\mathbf{A}$, we have $\mathrm{BRD}(\mathbf{A}, \mathbf{M}) = 1$.

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For Fraïssé classes, (Hjorth 2008) shows that $\mathrm{BRD}(\mathbf{A}, \mathcal{K}) \equiv 1$ implies $\mathrm{Aut}(\mathbf{K})$ trivial. So no "Big Ramsey Property."

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In particular, note that if $t < \omega$, $\mathrm{BRD}(\mathbf{A}, \mathbf{M}) \geq t$ iff there is an unavoidable t-coloring of $\mathrm{Emb}(\mathbf{A}, \mathbf{M})$.

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Enumerate $\mathbb{Q} = \{q_n : n < \omega\}$. Identify $[\mathbb{Q}]^2$ with $\mathrm{Emb}(2, \mathbb{Q})$. Given $q_m < q_n \in \mathbb{Q}$, color $\{q_m, q_n\}$ depending on whether $m < n$ or $n < m$.

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Fact (Exercise 1)

This 2-coloring of $\mathrm{Emb}(2, \mathbb{Q})$ is unavoidable.

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More informative than these numbers are the objects that they count, which are trees I like to call Devlin trees (see also *Joyce trees*)

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$\mathrm{BRD}(n, \mathbb{Q})$ = the number of Devlin trees with n coding nodes.

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- (Hubička 2020+) The class of finite posets has finite BRD. Uses Carlson-Simpson theorem. Forcing-free proof for triangle-free graphs.

- (Z. 2022): All finitely-constrained binary free amalgamation classes. These are classes in a finite relational language with only unary and binary symbols defined by forbidding a finite set of finite irreducible structures, e.g. finite k-clique-free graphs, finite directed graphs forbidding cyclic triangles, finite graphs with red/blue edges forbidding monochromatic triangles, etc. Generalized and streamlined Dobrinen's techniques by introducing aged coding trees.

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- (Balko-Chodounský-Dobrinen-Hubička-Konečný-Vena-Z. 2022+) Exact characterization of BRD for finitely-constrained binary free amalgamation classes. Again by defining suitable tree-like objects, but now with a third "interesting event," an age change.

Fix a Fraïssé class $\mathcal{K}$ with limit $\mathbf{K}$. Recall that given $\mathbf{A} \leq \mathbf{B} \in \mathcal{K}$ and finite colorings $\gamma_{\mathbf{A}}, \gamma_{\mathbf{B}}$ of $\mathrm{Emb}_{\mathbf{A}}, \mathrm{Emb}_{\mathbf{B}}$, respectively, we say $\gamma_{\mathbf{A}} \ll \gamma_{\mathbf{B}}$ iff whenever $f \in \mathrm{Emb}(\mathbf{A}, \mathbf{B})$ and $x, y \in \mathrm{Emb}_{\mathbf{B}}$ satisfy $\gamma_{\mathbf{B}}(x) = \gamma_{\mathbf{B}}(y)$, then $\gamma_{\mathbf{A}}(x \circ f) = \gamma_{\mathbf{A}}(y \circ f)$.

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Question: If each $\mathbf{A} \in \mathcal{K}$ has $\mathrm{BRD}(\mathbf{A}, \mathcal{K}) = t_{\mathbf{A}} < \omega$, then are there $\{\gamma_{\mathbf{A}} : \mathbf{A} \in [\mathbf{K}]^{<\omega}\}$ with each $\gamma_{\mathbf{A}}$ an unavoidable $t_{\mathbf{A}}$-coloring and with $\gamma_{\mathbf{A}} \ll \gamma_{\mathbf{B}}$ whenever $\mathbf{A} \leq \mathbf{B}$?

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Problem: Colorings in orbit closure of unavoidable $t_{\mathbf{A}}$-coloring need not be unavoidable $t_{\mathbf{A}}$-colorings!

When we can't prove something, just define it:

Definition (Z. 2019)

We say that $\mathcal{K}$ admits a big Ramsey structure if $\mathcal{K}$ has finite BRDs and there are $\{\gamma_{\mathbf{A}} : \mathbf{A} \in [\mathbf{K}]^{<\omega}\}$ with each $\gamma_{\mathbf{A}}$ an unavoidable BRD$(\mathbf{A}, \mathcal{K})$-coloring with $\gamma_{\mathbf{A}} \ll \gamma_{\mathbf{B}}$ when $\mathbf{A} \leq \mathbf{B}$.

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Question: Why? For Fraïssé classes, does finite BRDs imply existence of BRS?

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The dream: To each topological group G, assign a G-flow $\mathrm{D}(G)$ characterized by a suitable universal property up to isomorphism. When $G = \mathrm{Aut}(\mathbf{K})$, then $\mathrm{D}(G)$ should be metrizable iff $\mathcal{K}$ has finite big Ramsey degrees.

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Examples: For G locally compact, $\widehat{G} = G$. For $G = \mathrm{Aut}(\mathbf{K})$, we have $\widehat{G} \cong \mathrm{Emb}_{\mathbf{K}}$.

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Examples: All minimal flows, but can be non-minimal. For instance, the orbit closure of Devlin trees coding $\mathbb{Q}$. More generally, the orbit closure of any big Ramsey structure.

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1. Want something that works for finite BRD without necessarily assuming a BRS.
2. Unclear if universal completion flows exist for general topological groups.

Theorem (Z. 2019)

Suppose $G = \mathrm{Aut}(\mathbf{K})$ *and* $\mathcal{K}$ *admits a big Ramsey structure. Then there exists a universal completion flow, a completion flow which factors onto all others. This flow is unique up to isomorphism.*

Drawbacks:

1. Want something that works for finite BRD without necessarily assuming a BRS.
2. Unclear if universal completion flows exist for general topological groups.
3. Even if G has a universal completion flow, metrizability is used in the proof of uniqueness.

Second attempt: Note that we can identify $\widehat{G} \subseteq \mathrm{Sa}(G)$ with $\{\mathsf{p} \in \mathrm{Sa}(G) : \forall U \in \mathcal{N}_G \ \exists g \in G \ (gU \in \mathsf{p})\}$.

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The compact left-topological semigroup structure on $\widehat{G}$ gives us a left action of $\widehat{G}$ on $\mathrm{Sa}(G)$ which is not jointly continuous, but for each $\eta \in \widehat{G}$, $\lambda_\eta \colon \mathrm{Sa}(G) \to \mathrm{Sa}(G)$ is an injective G-map. Write $\eta{\cdot}\mathrm{Sa}(G)$ for the image G-flow.

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We consider subflows $X \subseteq \mathrm{Sa}(G)$ minimal with respect to the property that for some net $(\eta_i)_{i\in I}$ from $\mathrm{Sa}(G)$, $X = \lim_i \eta_i{\cdot}\mathrm{Sa}(G)$ in the Vietoris topology. In BRS case, this recovers the universal completion flow.

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Problem: Such X likely not unique up to isomorphism in general. But they are unique up to weak equivalence.

Definition (Z. 2024+)

Given a topological group G and G-flows X and Y, we say that$X$ is weakly contained in Y if there are a G-flow Z and a net $(X_i)_{i \in I}$ of subflows of Z, all $X_i \cong X$, and $\lim_i X_i$ exists and is isomorphic to Y. Weak equivalence is then just weak containments in each direction.

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Subtlety: Not at all clear that weak containment is a pre-order or that weak equivalence is an equivalence relation! Exhibiting large families of topological groups and G-flows for which this holds takes significant work.

Thanks!