Big Ramsey degrees, structures, dynamics I

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Outline of part I.

- Introduction to topological dynamics.

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- Construction of the Samuel compactification and the universal minimal flow – with lots of exercises :)

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- Construction of the Samuel compactification and the universal minimal flow – with lots of exercises :)
- First-order structures and Fraïssé classes
- The Samuel compactification of $\mathrm{Aut}(\mathbf{K})$.

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A right G-flow is a compact (Hausdorff) space and a continuous map $X \times G \to X$ satisfying $x \cdot e_G = x$ and $(x \cdot g) \cdot h = x \cdot gh$ whenever $x \in X$ and $g, h \in G$.

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If X, Y are G-flows, a G-map $\pi \colon X \to Y$ is a map which is continuous and G-equivariant, i.e. $\pi(x \cdot g) = \pi(x) \cdot g$. We call π a factor map if it is surjective.

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A subflow of X is a non-empty, closed, G-invariant subspace. An orbit of X is a subset of the form $x \cdot G := \{xg : g \in G\}$ for some $x \in X$.

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Fact (Ellis 1960)

Every topological group admits a universal minimal flow, a minimal G-flow which factors onto every other minimal G-flow. Such a flow is unique up to isomorphism, and denoted $\mathrm{M}(G)$.

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Given $\mathsf{p} \subseteq \mathcal{P}(G)$, we say that p has the near finite intersection property (NFIP) if whenever $F \in [\mathsf{p}]^{<\omega}$, and $U \in \mathcal{N}_G$, we have $\bigcap_{S \in F} SU \neq \emptyset$. We call p a near ultrafilter on G if it is maximal under inclusion with respect to having the NFIP.

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Let $\mathrm{Sa}(G)$ denote the set of near ultrafilters on G. Note that for G discrete, we have $\mathrm{Sa}(G) = \beta G$, the set of ultrafilters on G.

Fact (Exercise 1)

Fix $\mathsf{p} \in \mathrm{Sa}(G)$.

1. If $S \subseteq G$ and $S \notin \mathsf{p}$, then there is $U \in \mathcal{N}_G$ with $SU \notin \mathsf{p}$.

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1. If $S \subseteq G$ and $S \notin \mathsf{p}$, then there is $U \in \mathcal{N}_G$ with $SU \notin \mathsf{p}$.
2. If $S \in \mathsf{p}$, $n < \omega$, and $S_0, ..., S_{n-1} \subseteq S$ with $\bigcup_{i<n} S_i$ dense in S, then for some $i < n$, $S_i \in \mathsf{p}$.

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Given $S \subseteq G$, let $C_S = \{\mathsf{p} \in \mathrm{Sa}(G) : S \in \mathsf{p}\}$, $N_S = \mathrm{Sa}(G) \setminus C_S$. Note $N_S \cap N_T = N_{S \cup T}$. Equip $\mathrm{Sa}(G)$ with the topology given by basis $\{N_S : S \subseteq G \text{ not dense}\}$.

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Fact (Exercise 2)

This topology is compact Hausdorff.

G acts on $\mathrm{Sa}(G)$ as expected; if $\mathsf{p} \in \mathrm{Sa}(G)$, $g \in G$, and $S \subseteq G$, we set $S \in \mathsf{p}g$ iff $Sg^{-1} \in \mathsf{p}$.

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Fact (Exercise 4)

The map $g \to \tilde{g}$ is a homeomorphism onto its image. Hence we drop the $\tilde{g}$ notation and simply identify $G \subseteq \mathrm{Sa}(G)$.

Theorem

Whenever X is a G-flow and $x \in X$, there is a (unique) G-map $\lambda_x \colon \mathrm{Sa}(G) \to X$ satisfying $\lambda_x(e_G) = x$.

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For continuity, if $A \subseteq X$ is open, then

$$\lambda_x^{-1}(A) = \bigcup \{N_{\{g \in G \colon xg \in X \setminus B\}} : B \subseteq X \text{ open and } \overline{B} \subseteq A\}.$$

λ_x is clearly G-equivariant. □

To build a universal minimal flow, let $M \subseteq \mathrm{Sa}(G)$ be any minimal subflow. If X is a minimal G-flow, pick any $x \in X$. Then $\lambda_x|_M : M \to X$ is a G-map. As X is minimal, it must be onto, i.e. a factor map.

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Uniqueness of $\mathrm{M}(G)$ is a bit trickier. The classical proof involves endowing $\mathrm{Sa}(G)$ with the structure of a compact left-topological semigroup to show that any minimal $M \subseteq \mathrm{Sa}(G)$ is coalescent, meaning that any G-map from M to M is an isomorphism.

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(Gutman-Li 2013) provides a direct argument that any universal minimal flow M is coalescent (idea: if not, build a really long inverse limit, using universality to keep construction going. This inverse limit is still minimal, but has too large cardinality).

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An $\mathcal{L}$-structure $\mathbf{A} = \langle A, (R_i^{\mathbf{A}})_{i \in I} \rangle$ is a set A and for each $i \in I$, a distinguished subset $R_i^{\mathbf{A}} \subseteq A^{n_i}$. Typically denote structures in **bold** and use the un-bold letter for the underlying set.

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If $\mathbf{A}, \mathbf{B}$ are $\mathcal{L}$-structures, an embedding of $\mathbf{A}$ into $\mathbf{B}$ is a map $f\colon A \to B$ such that for each $i \in I$ and $(a_0, ..., a_{n_i-1}) = \in A^{n_i}$, we have $(a_0, ..., a_{n_i-1}) \in R_i^{\mathbf{A}} \Leftrightarrow (f(a_0), ..., f(a_{n_i-1})) \in R_i^{\mathbf{B}}$.
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$\mathrm{Emb}(\mathbf{A}, \mathbf{B})$ – embeddings $\mathbf{A}$ to $\mathbf{B}$.

Surjective embeddings are called isomorphisms, and an isomorphism from $\mathbf{A}$ to $\mathbf{A}$ is an automorphism. Write $\mathrm{Aut}(\mathbf{A})$ for the group of automorphisms of $\mathbf{A}$. $\mathbf{A}$ is a substructure of $\mathbf{B}$ if $A \subseteq B$ and the inclusion is an embedding.

Write $\mathrm{Fin}(\mathcal{L})$ for the class of finite $\mathcal{L}$-structures. Fix a countable $\mathcal{L}$ and a countably infinite $\mathcal{L}$-structure $\mathbf{K}$. We set $\mathrm{Age}(\mathbf{K}) = \{\mathbf{A} \in \mathrm{Fin}(\mathcal{L}) : \mathrm{Emb}(\mathbf{A}, \mathbf{K}) \neq \emptyset\}$.

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Definition

We say that $\mathbf{K}$ is ultrahomogeneous if whenever $\mathbf{A} \in \mathrm{Age}(\mathbf{K})$ and $f_0, f_1 \in \mathrm{Emb}(\mathbf{A}, \mathbf{K})$, there is $g \in \mathrm{Aut}(\mathbf{K})$ with $g \circ f_0 = f_1$.

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A weaker-looking, but equivalent property:

Fact (Exercise 5)

$\mathbf{K}$ is ultrahomogeneous iff $\mathbf{K}$ has the extension property (ExtP): whenever whenever $\mathbf{A} \subseteq \mathbf{B} \in \mathrm{Age}(\mathbf{K})$ and $f \in \mathrm{Emb}(\mathbf{A}, \mathbf{K})$, there is $g \in \mathrm{Emb}(\mathbf{B}, \mathbf{K})$ with $g|_{\mathbf{A}} = f$.

Definition

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- Joint Embedding Property: If $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, then there is $\mathbf{C} \in \mathcal{K}$ with both $\mathrm{Emb}(\mathbf{A}, \mathbf{C})$ and $\mathrm{Emb}(\mathbf{B}, \mathbf{C})$ non-empty.

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- Amalgamation Property: If $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$, $f \in \mathrm{Emb}(\mathbf{A}, \mathbf{B})$, and $g \in \mathrm{Emb}(\mathbf{A}, \mathbf{C})$, then there are $\mathbf{D} \in \mathcal{K}$, $r \in \mathrm{Emb}(\mathbf{B}, \mathbf{D})$, and $s \in \mathrm{Emb}(\mathbf{C}, \mathbf{D})$ with $r \circ f = s \circ g$.

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Fact (Exercise 6)

If $\mathbf{K}$ is a countably infinite, ultrahomogeneous $\mathcal{L}$-structure, then $\mathrm{Age}(\mathbf{K})$ is a Fraïssé class. Call such $\mathbf{K}$ Fraïssé structures.

Theorem (Fraïssé 1954)

If $\mathcal{K}$ is a Fraïssé class, then there is a Fraïssé structure $\mathbf{K}$ such that $\mathrm{Age}(\mathbf{K}) = \mathcal{K}$. Such a $\mathbf{K}$ is unique up to isomorphism.

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From now on, fix a Fraïssé structure $\mathbf{K}$. Given $\mathbf{A} \in \mathrm{Age}(\mathbf{K})$, write $\mathrm{Emb}_{\mathbf{A}}$ for $\mathrm{Emb}(\mathbf{A}, \mathbf{K})$.
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$[\mathbf{K}]^{<\omega} :=$ finite substructures of $\mathbf{K}$.

View $G := \mathrm{Aut}(\mathbf{K})$ as a topological group by setting $\mathcal{N}_G = \{\mathrm{Stab}(\mathbf{A}) : \mathbf{A} \in [\mathbf{K}]^{<\omega}\}$. Sometimes called the pointwise convergence topology. Write $U_{\mathbf{A}}$ for $\mathrm{Stab}(\mathbf{A})$

Which topological groups have the form $\mathrm{Aut}(\mathbf{K})$ for a Fraïssé structure $\mathbf{K}$? These are all closed subgroups of S_∞, the group of all permutations of ω with the pointwise convergence topology.

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Fact (Exercise 7)

If G is a closed subgroup of S_∞, then there is a countable relational language $\mathcal{L}$ and a Fraïssé $\mathcal{L}$-structure on underlying set ω such that $G = \mathrm{Aut}(\mathbf{K})$.

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By a result of (Becker-Kechris 1994), the topological groups isomorphic to closed subgroups of S_∞ are exactly those Polish groups which are non-Archimedean, i.e. admit a base at e_G of open subgroups.

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In particular, whenever $\mathsf{p} \in \mathrm{Sa}(G)$, we have $\mathsf{p}|_{\mathbf{A}} := \{SU_{\mathbf{A}} : S \in \mathsf{p}\} \in \beta\mathrm{Emb}_{\mathbf{A}}$. Hence we can identify

$$\mathrm{Sa}(G) \cong \varprojlim \beta\mathrm{Emb}_{\mathbf{A}}.$$

The inverse limit is with respect to the natural maps $\beta\mathrm{Emb}_{\mathbf{B}} \to \beta\mathrm{Emb}_{\mathbf{A}}$ whenever $\mathbf{A} \subseteq \mathbf{B} \in [\mathbf{K}]^{<\omega}$.

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Given $\mathsf{p} \in \mathrm{Sa}(G)$, $g \in G$, $\mathbf{A} \in [\mathbf{K}]^{<\omega}$, and $S \subseteq \mathrm{Emb}_{\mathbf{A}}$, we have $S \in (\mathsf{p}g)|_{\mathbf{A}}$ iff $\{f \circ g^{-1} : f \in S\} \in \mathsf{p}|_{g\mathbf{A}}$.