P-points and related ultrafilters — part III

Juris Steprāns

York University

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The goal of this last lecture is to explain some techniques of Shelah [3] for destroying some P-points while preserving others. Selective ultrafilters and the games considered in the first lecture will play a key role. Of course, it is not possible to preserve a single ultrafilter, but only an equivalence class of ultrafilters. The following definition will be used soon and makes this precise.

Definition 1

If $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters define $\mathcal{U} \equiv_{\mathsf{RK}} \mathcal{V}$ if there is a bijection ψ such that $A \in \mathcal{V}$ if and only if $\psi^{-1}(A) \in \mathcal{U}$. Define $\mathcal{U} \leq_{\mathsf{RK}} \mathcal{V}$ if there is a function ψ such that $A \in \mathcal{U}$ if and only if $\psi^{-1}(A) \in \mathcal{V}$.

It is a nice exercise to show that if $\mathcal{U} \leq_{\mathsf{RK}} \mathcal{V}$ and $\mathcal{V} \leq_{\mathsf{RK}} \mathcal{U}$ then $\mathcal{U} \equiv_{\mathsf{RK}} \mathcal{V}$.

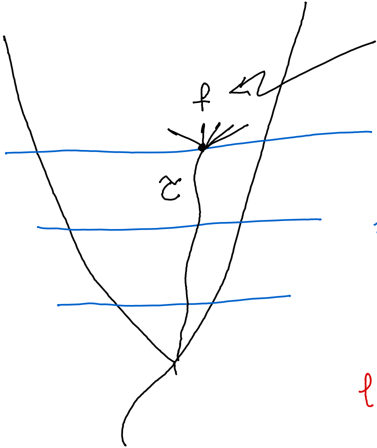
Definition 2

Given an ultrafilter $\mathcal{U}$ *define the partial order* $\mathbb{P}(\mathcal{U})$ *to consist of all trees* $\mathbb{T}$ *such that* $\mathbf{succ}_{\mathbb{T}}(\tau) \subseteq 2^{|\tau|}$ *and for which there is* $U \in \mathcal{U}$ *such that*

$$(\forall \ell \in \omega)(\forall^\infty k \in U)(\forall t \in Lev_k(\mathbb{T}))(\forall h : \ell \to 2) \\ (\exists f \in \mathbf{succ}_{\mathbb{T}}(t))\ h \subseteq f. \quad (1)$$

The ordering on $\mathbb{P}(\mathcal{U})$ *is inclusion.*

If $G \subseteq \mathbb{P}(\mathcal{U})$ *is generic then define* B_G *by* $B_G(k) = f$ *if and only if for every* $\mathbb{T} \in G$ *there is* $t \in \mathbb{T}$ *such that* $t(k) = f$*. Define a colouring* $C_G : [\omega]^2 \to 2$ *by* $\mathbb{C}_G(a) = B_G(\max(a))(\min(a))$.



$f \in \mathrm{succ}_T(\tau)$

$h \subseteq f$

$k \in U \in \mathcal{U}$

$\ell \quad T \xrightarrow{h} 2$

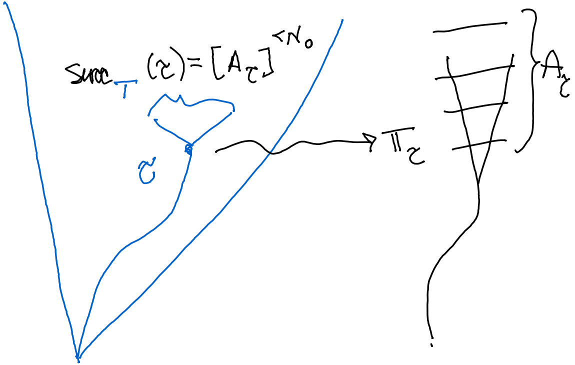
Lemma 1

If $\mathcal{U}$ is a P-point then $\mathbb{P}(\mathcal{U})$ is proper and ω^ω bounding.

Proof.

Given $\mathbb{T} \in \mathbb{P}(\mathcal{U})$ and $\{D_n\}_{n\in\omega}$ that are dense subsets of $\mathbb{P}(\mathcal{U})$ construct a $\mathcal{U}$-P-tree T such that for each $\tau \in T$ there is $\mathbb{T}_\tau \in \mathbb{P}(\mathcal{U})$ and $A_\tau \in \mathcal{U}$ such that:

- $\mathbb{T}_\varnothing = \mathbb{T}$
- $(\forall k \in A_\tau)(\forall t \in Lev_k(\mathbb{T}_\tau))(\forall h : |\tau| \to 2)(\exists f \in \mathbf{succ}_{\mathbb{T}}(t))\ h \subseteq f$
- $\mathbf{succ}_T(\tau) = [A_\tau]^{<\aleph_0}$
- if $\tau \subseteq \sigma$ and $|\tau| = n+1$ and $k = \max(\tau(n))$ then $\mathbb{T}_\sigma \subseteq \mathbb{T}_\tau$ and $Lev_k(\mathbb{T}_\tau) = Lev_k(\mathbb{T}_\sigma)$
- if $|\tau| = n+1$ and $k = \max(\tau(n))$ and $t \in Lev_k(\mathbb{T}_\tau)$ then $\mathbb{T}\langle t\rangle \in D_n$.


$\mathrm{Succ}_T(\tau) = [A_\tau]^{<\aleph_0}$
τ
T_τ
A_τ

Proof.

Since T is a $\mathcal{U}$-P-tree let B be a branch of T such that $\bigcup_n B(n) \in \mathcal{U}$ and let

$$\mathbb{T}^* = \bigcup_n Lev_{B(n)}\left(\mathbb{T}_{B\restriction(n+1)}\right).$$

It is routine to check that $\mathbb{T}^* \in \mathbb{P}(\mathcal{U})$ and it has the desired properties.

$\square$

Definition 3

Given $P : [\omega]^2 \to 2$ a set $X \subseteq \omega$ will be said to be **almost-J-homogeneous for** *P if for all $x \in X$ there are only finitely many $y \in X$ such that $P(x, y) \neq J$.*

Lemma 2

If $\mathcal{U}$ is a P-point and $P : [\omega]^2 \to 2$ then there is $J \in 2$ and a set $X \in \mathcal{U}$ that is almost-J-homogeneous for P.

Proof.

It is an exercise to see the same proof as for selective ultrafilters works. □

Lemma 3

If $\mathcal{U}$ is a P-point and $J \in 2$ and $\mathbb{Q}$ is a $\mathbb{P}(\mathcal{U})$ name for a partial order such that $\mathbf{1} \Vdash_{\mathbb{P}(\mathcal{U})*\mathbb{Q}}$ "$\mathbb{Q}$ is ω^ω bounding" and

$$\mathbf{1} \Vdash_{\mathbb{P}(\mathcal{U})*\mathbb{Q}} \text{“}\dot{X} \text{ is almost-}J\text{-homogeneous for } C_{\dot{G}}\text{”}$$

then there is $\mathbb{T} \in \mathbb{P}(\mathcal{U})$ and $A \in \mathcal{U}$ such that

$$\mathbb{T} \Vdash_{\mathbb{P}(\mathcal{U})*\mathbb{Q}} \text{“}A \cap \dot{X} = \varnothing\text{”}.$$

Proof.

Assume that J, the almost homogeneous colour for $\dot{X}$, is 0. If it happens that $\mathbf{1} \not\Vdash_{\mathbb{P}(\mathcal{U})*\mathbb{Q}}$ "$|\dot{X}| = \aleph_0$" then the result is immediate, so let $\dot{\psi}$ be a $\mathbb{P}(\mathcal{U}) * \mathbb{Q}$ name such that

$$\mathbf{1} \Vdash_{\mathbb{P}(\mathcal{U})*\mathbb{Q}} \text{“}(\forall k \in \omega)(\exists m \in \dot{X} \setminus k)(\forall \ell \in \dot{X}) \text{ if } C_{\dot{G}}(m, \ell) = 1 \text{ then } \ell < \dot{\psi}(k)\text{”}. \quad (2)$$

Since $\mathbb{P}(\mathcal{U}) * \mathbb{Q}$ is ω^ω-bounding by Lemma 1 it is possible to find $\mathbb{T}$ and $\Psi : \omega \to \omega$ such that

$$\mathbb{T} \Vdash_{\mathbb{P}(\mathcal{U})*\mathbb{Q}} \text{“}(\forall k \in \omega)\ \dot{\psi}(k) \leq \Psi(k)\text{”}.$$

Continuation of proof.

Find A such that:

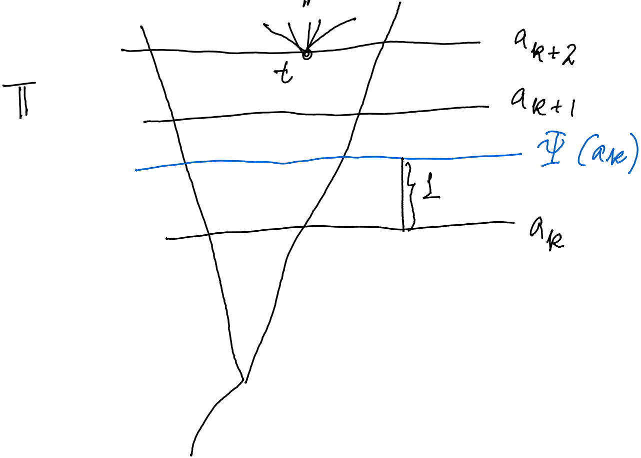
- $A \in \mathcal{U}$ and A is enumerated in order by $\{a_i\}_{i\in\omega}$
- A witnesses that $\mathbb{T} \in \mathbb{P}(\mathcal{U})$ in the strong sense that if $t \in \mathrm{Lev}_{a_{n+1}}\mathbb{T}$ and $h : a_n \to 2$, then there is $f \in \mathbf{succ}_{\mathbb{T}}(t)$ such that $h \subseteq f$
- $\Psi(a_n) < a_{n+1}$ for all n.

For $t \in \mathrm{Lev}_{a_{i+2}}(\mathbb{T})$ let

$$\mathcal{S}(t) = \{f \in \mathbf{succ}_{\mathbb{T}}(t) \mid (\forall x \in [a_i, \Psi(a_i)))\ f(x) = 1\}$$

and note that follows that if $t \in \mathrm{Lev}_{a_{i+2}}(\mathbb{T})$ and $h : a_i \to 2$ then there is $f \in \mathbf{succ}_{\mathbb{T}}(t)$ such that $h \subseteq f$ and $f(\ell) = 1$ if $a_i \leq \ell < a_{i+1}$. Since $\Psi(a_i) < a_{i+1}$ it follows that $f \in \mathcal{S}(t)$.

$$S = \{f \in \mathrm{succ}_{\mathbb{T}}(t) \mid f \restriction [a_k, \Psi(a_k)) \equiv 1\}$$



Continuation of proof.

Therefore if $\mathbb{T}^*$ is defined by

$$\mathbb{T}^* = \bigcap_{i\in\omega}\left(\bigcup_{t\in\mathrm{Lev}_{a_{i+2}}(\mathbb{T})}\ \bigcup_{f\in\mathcal{S}(t)} \mathbb{T}\langle t^\frown f\rangle\right)$$

then $\mathbf{succ}_{\mathbb{T}^*}(t) = \mathcal{S}(t)$ for each $i \in \omega$ and $t \in \mathrm{Lev}_{a_{i+2}}(\mathbb{T})$. It follows that A witnesses that $\mathbb{T}^* \in \mathbb{P}(\mathcal{U})$.

Continuation of proof.

Finally, it suffices to show that if $k > 0$ then $\mathbb{T}^* \Vdash_{\mathbb{P}(\mathcal{U})}$ "$a_k \notin \dot{X}$". In order to establish this, note that

$$\mathbb{T}^* \Vdash \text{"}(\exists x \in \dot{X} \cap [a_{k-1}, \Psi(a_{k-1}))(\forall y \in \dot{X} \setminus \Psi(a_{k-1}))\ P(x, y) = 0\text{"}.$$

but this contradicts that if $t \in \mathrm{Lev}_{a_k}(\mathbb{T}^*)$ and $f \in \mathbf{succ}_{\mathbb{T}^*}(t)$ then $f \in \mathcal{S}(t)$ and so $f(\{x, a_k\}) = 1$ for all $x \in [a_{k-1}, \Psi(a_{k-1})]$.

This is exactly what is required since then

$$T^* \Vdash_{\mathbb{P}(\mathcal{U})} \text{"}\Psi(a_{k-1}) < a_k \\ \& \ (\forall x \in \dot{X} \cap [a_{k-1}, \Psi(a_{k-1}))\ P(x, a_k) = 1\text{"}. \quad (3)$$

Corollary 1

If $\mathcal{U}$ is a P-point and $\mathbb{Q}$ is ω^ω-bounding then $\mathbb{P}(\mathcal{U}) * \mathbb{Q}$ does not preserve $\mathcal{U}$.

Proof.

If $\mathcal{U}$ is a P-point then Lemma 1 establishes that and $\mathbb{P}(\mathcal{U})$ is proper and ω^ω bounding. One the other hand, it follows from Lemma 3 and Lemma 2 that $\mathcal{U}$ is not a P-point after forcing with $\mathbb{P}(\mathcal{U}) * \mathbb{Q}$. $\square$

Using the corollary, countable support iteration over a model of $\diamondsuit_{\omega_2}$ and standard forcing theorems produces a third model with no P-points. But our current goal is to get a model with a single P-point (up to RK equivalence).

Definition 4

Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters on ω. Say that T is a $(\mathcal{U}, \mathcal{V})$-SP-tree if for each $\tau \in T$ if

- *τ is even then there is $A \in \mathcal{U}$ such that $\mathbf{succ}_T(\tau) = A$*
- *if τ is odd then there is $A \in \mathcal{V}$ such that $\mathbf{succ}_T(\tau) = [A]^{<\aleph_0}$*
- *$\min(A) > \tau(\ell)$ for all ℓ in the domain of τ.*

("P" is for P-point and "S" is for selective.)

Lemma 4

Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters. The following are then equivalent:

1. *$\mathcal{U}$ is selective and $\mathcal{V}$ is a P-point and $\mathcal{U} \not\leq_{\mathrm{RK}} \mathcal{V}$*
2. *Every $(\mathcal{U}, \mathcal{V})$-SP-tree has a branch B such that*
    - $\bigcup_{n\in\omega} B(2n+1) \in \mathcal{V}$
    - $\{B(2n) \mid n \in \omega\} \in \mathcal{U}$.

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Peoof. Jump to applying Lemma 4.

To see that (2) implies (1) note first that (2) implies that $\mathcal{U}$-S-tree has a branch with range in $\mathcal{U}$ and so $\mathcal{U}$ is selective. It also follows from (2) that $\mathcal{V}$-P-tree has a branch B such that $\bigcup_n B(n) \in \mathcal{V}$ and so $\mathcal{V}$ is a P-point.

To see that $\mathcal{U} \not\leq_{\mathrm{RK}} \mathcal{V}$ suppose that $F : \omega \to \omega$ witnesses that $\mathcal{U} \leq_{\mathrm{RK}} \mathcal{V}$. Let T be the $(\mathcal{V},\mathcal{U})$-PS-tree such that:

- if $\tau \in T$ and $|\tau| = 2n$ is even then $\mathbf{succ}_T(\tau) = \omega \setminus F(\bigcup_{m\in n} \tau(2m+1))$
- if $\tau \in T$ and $|\tau| = 2n+1$ is even then $\mathbf{succ}_T(\tau) = [\omega \setminus \bigcup_{m\leq n} F^{-1}(\tau(2m))]^{<\aleph_0}$

It follows that if B is a branch of T it must be the case that

$$F^{-1}(\{B(2k)\}_{k\in\omega}) \cap \bigcup_{k\in\omega} B(2k+1) = \varnothing$$

and so either $\{B(2k)\}_{k\in\omega} \notin \mathcal{U}$ or $\bigcup_{k\in\omega} B(2k+1) \notin \mathcal{V}$.

Second part of proof.

To see that (1) implies (2) let T be a $(\mathcal{U}, \mathcal{V})$-SP-tree. For each $\tau \in T$ such that $|\tau|$ is odd let $W_\tau \in \mathcal{V}$ be such that $\mathbf{succ}_T(\tau) = [W_\tau]^{<\aleph_0}$ and then find $W \in \mathcal{V}$ such that $W \subseteq * W_\tau$ for each $\tau \in T$ with $|\tau|$ odd.

Now define the partition $[\omega]^4 = P_0 \cup P_1$ by $\{\ell_0, \ell_1, \ell_2, \ell_3\} \in P_0$ if $\ell_3 \in \mathbf{succ}_T(\tau \restriction (2n+2))$ for every $\tau \in T$ for which there is $n \in \omega$ such that

1. $\tau(2n) = \ell_0$
2. $\tau(2n+1) = W \cap [\ell_1, \ell_2) \subseteq W_{\tau \restriction 2n+1}$.

Use that $\mathcal{U}$ is selective find $Y \in \mathcal{U}$ and $J \in 2$ such that $[Y]^4 \subseteq P_J$.

Continuation of proof.

The first thing to observe is that $J = 0$. To see this let $\ell_0 \in Y$ and let

$$\mathcal{T} = \{\tau \in T \mid (\exists n)\ \tau(2n) = \ell_0 \ \&\ |\tau| = 2n+1\}$$

and then let $M > \ell_0$ be so large that $W \setminus M \subseteq W_\tau$ for all τ in the finite set $\mathcal{T}$. Then let $\ell_1 \in Y$ and $\ell_2 \in Y$ be such that $M < \ell_1 < \ell_2$. Let

$$\ell_3 \in Y \cap \bigcap_{\tau \in \mathcal{T}} \mathbf{succ}_T(\tau^\frown(W \cap [\ell_1, \ell_2))).$$

Note that $W \cap [\ell_1, \ell_2) \in [W_\tau]^{\aleph_0}$ for each $\tau \in T$ and so $\mathbf{succ}_T(\tau^\frown(W \cap [\ell_1, \ell_2)))$ is defined. Hence $\{\ell_0, \ell_1, \ell_2, \ell_3\} \in P_0$ and so $J = 0$.

Continuation of proof.

Let Y be enumerated in order as $\{y_i\}_{i\in\omega}$. Consider first the case that for every $Z \subseteq \omega$

$$\bigcup_{i\in Z}[y_{i-1}, y_{i+1}) \in \mathcal{V} \quad \textbf{if} \quad \{y_i\}_{i\in Z} \in \mathcal{U}. \tag{4}$$

Since $Y \in \mathcal{V}$ it follows that for some $J \in 3$ it must be the case $\{y_{3i+J}\}_{i\geq 1} \in \mathcal{V}$ and hence $\bigcup_{i\geq 1}[y_{3i+J-1}, y_{3i+J+1}) \in \mathcal{V}$. To simplify notation, there is no harm in assuming that $J = 0$. Then the mapping

$$F : \bigcup_{i\geq 1}[y_{3i-1}, y_{3i+1}) \to \{y_{3i}\}_{i\geq 1}$$

defined by $F(k) = y_{3i}$ if and only if $y_{3i-1} \leq k < y_{3i+1}$ witnesses that $\mathcal{U} \leq_{\mathrm{RK}} \mathcal{V}$ and there is nothing more to do.

Continuation of proof.

Hence, it can be assumed that there is some $Z \subseteq \omega$ such that (4) fails. Let $\{z(i)\}_{i\in\omega}$ enumerate Z in order so that

$$\{y_{z(i)}\}_{i\in\omega} \in \mathcal{U} \quad \textbf{and} \quad \bigcup_{i\in\omega}[y_{z(i)-1}, y_{z(i)+1}) \notin \mathcal{V}.$$

In other words, $\bigcup_{i\in\omega}[y_{z(i)+1}, y_{z(i+1)-1}) \in \mathcal{V}$ and it follows that

$$D = W \cap \bigcup_{i\in\omega}([y_{z(i)+1}, y_{z(i+1)-1}) \in \mathcal{V}.$$

Continuation of proof.

Let B be defined for $i \in \omega$ by

$$B(2i) = y_{z(i)} \;\;\&\;\; B(2i+1) = W \cap [y_{z(i)+1}, y_{z(i+1)-1}).$$

Then $\{B(2i)\}_{i\in\omega} = \{y_{z(i)}\}_{i\in\omega} \in \mathcal{U}$ and

$$\bigcup_{i\in\omega} B(2i+1) = \bigcup_{i\in\omega} W \cap [y_{z(i)+1}, y_{z(i+1)-1}) = D \in \mathcal{V}$$

and so it suffices to show that $B \restriction k \in T$ for all k.

To see that this is so use that Z is P_0-homogeneous.

Continuation of proof.

By dropping finitely many elements of Z it may be assumed that $y_{z(0)} \in \mathbf{succ}_T(\varnothing)$. Now suppose that $B \restriction 2n \in T$ and that $y(z(n)) \in \mathbf{succ}_T(B \restriction 2n)$. (This holds with $n = 0$.) Then $\{y_{z(n)}, y_{z(n)+1}, y_{z(n+1)-1}, y_{z(n+1)}\} \in P_0$ and so

$$W \cap [y_{z(n)+1}, y_{z(n+1)-1}) \subseteq W_{B \restriction 2n+1}$$

and so

$$B \restriction (2n+2) = (B \restriction 2n+1)^\frown W \cap [y_{z(n)+1}, y_{z(n+1)-1}) \in T$$

and so $y_{z(n+1)} \in \mathbf{succ}_T(B \restriction (2n+2))$ and so $B \restriction (2n+3) \in T$ as required to continue the induction.

□

Notation 1

If $\mathcal{U}$ is an ultrafilter, $\mathbb{P}$ is a partial order and $\dot{A}$ a $\mathbb{P}$-name such that $1 \Vdash_{\mathbb{P}}$ "$\dot{A} \subseteq \omega$" then let $D(\dot{A}, \mathcal{U}, \mathbb{P})$ denote the set

$$\left\{ r \in \mathbb{P} \;\middle|\; (\exists Z \in \mathcal{U})(\forall n \in \omega)(\exists r_n \leq r)\ r_n \Vdash_{\mathbb{P}} \text{“} Z \cap n \subseteq \dot{A} \cap n \text{”} \right\}. \tag{5}$$

Lemma 5

If $\mathcal{U}$ is an ultrafilter and $\mathbb{P}$ a partial order and $\dot{A}$ a $\mathbb{P}$-name such that $1 \Vdash_{\mathbb{P}}$ "$\dot{A} \subseteq \omega$" then

$$D(\dot{A}, \mathcal{U}, \mathbb{P}) \cup D(\omega \setminus \dot{A}, \mathcal{U}, \mathbb{P}) = \mathbb{P}.$$

This can be proved using a fake generic.

Lemma 6

If $\mathcal{U}$ is selective and $\mathcal{V}$ is a P-point and $\mathcal{U} \not\leq_{\mathsf{RK}} \mathcal{V}$ then forcing with $\mathbb{P}(\mathcal{V})$ preserves $\mathcal{U}$.

Proof.

By Lemma 4 the hypothesis implies that for every $(\mathcal{U},\mathcal{V})$-SP-tree has a branch B such that

- $\bigcup_{n\in\omega} B(2n+1) \in \mathcal{V}$
- $\{B(2n) \mid n \in \omega\} \in \mathcal{U}$.

It will be shown that if $\mathbf{1} \Vdash_{\mathbb{P}(\mathcal{V})}$ "$\dot{X} \subseteq \omega$" then there is some $A \in \mathcal{U}$ and $\mathbb{T} \in \mathbb{P}(\mathcal{V})$ such that either $\mathbb{T} \Vdash_{\mathbb{P}(\mathcal{V})}$ "$\dot{X} \supseteq A$" or $\mathbb{T} \Vdash_{\mathbb{P}(\mathcal{V})}$ "$\dot{X} \cap A = \varnothing$". Using Lemma 5 and Notation 1 it is possible to find $\mathbb{T} \in \mathbb{P}(\mathcal{V})$ such that either $D(\dot{X},\mathcal{U},\mathbb{P}(\mathcal{V}))$ or $D(\omega \setminus \dot{X},\mathcal{U},\mathbb{P}(\mathcal{V}))$ is dense below $\mathbb{T}$; without loss of generality, assume that $D(\dot{X},\mathcal{U},\mathbb{P}(\mathcal{V}))$ is dense below $\mathbb{T}$.

Proof.

Construct a $(\mathcal{U},\mathcal{V})$-SP-tree T such that for each $\tau \in T$ there is $\mathbb{T}_\tau \in \mathbb{P}(\mathcal{V})$ and A_τ such that

1. $\mathbb{T}_\varnothing = \mathbb{T}$
2. if $|\tau|$ is even then $\mathbf{succ}_{\mathbb{T}}(\tau) = A_\tau \in \mathcal{U}$
3. if $|\tau|$ is odd then $\mathbf{succ}_{\mathbb{T}}(\tau) = [A_\tau]^{<\aleph_0}$ and $A_\tau \in \mathcal{V}$
4. if $\tau \subseteq \sigma$ and $|\tau| = n+1$ and $k = \max(\tau(n))$ then $\mathbb{T}_\sigma \subseteq \mathbb{T}_\tau$ and $Lev_k(\mathbb{T}_\tau) = Lev_k(\mathbb{T}_\sigma)$
5. if $|\tau|$ is odd and $k \in A_\tau$ then for all $t \in Lev_k(\mathbb{T}_\tau))(\forall h : |\tau| \to 2)(\exists f \in \mathbf{succ}_{\mathbb{T}}(t))\ h \subseteq f$
6. if $|\tau|$ is even and $k \in A_\tau$ then $\mathbb{T}_{\tau^\frown k} \Vdash_{\mathbb{P}(\mathcal{V})}$ "$k \in \dot{X}$".

Proof.

This is an induction similar to the proof of properness. For example, to see that (6) holds let $|\tau| = 2n$ and suppose that $\mathbb{T}_\tau$ is given. For $k = \max(\tau(2n-1))$ and $t \in \mathrm{Lev}_k(\mathbb{T})$ use that $D(\dot{X}, \mathcal{U}, \mathbb{P}(\mathcal{V}))$ is dense below $\mathbb{T}$ to find $A^*_t \in \mathcal{U}$ and a sequence $\{\mathbb{T}_{t,n}\}_{n \in A^*_t}$ such that $\mathbb{T}_{t,n} \Vdash_{\mathbb{P}(\mathcal{V})}$ "$n \in \dot{X}$" for each $n \in A^*_t$. Then let

$$A_\tau = \bigcap_{t \in \mathrm{Lev}_k(\mathbb{T})} A^*_t$$

and for each $n \in A^*_t$

$$\mathbb{T}_{\tau^\frown n} = \bigcup_{t \in \mathrm{Lev}_k(\mathbb{T})} \mathbb{T}_{t,n}.$$

Proof.

Then T is a $(\mathcal{U},\mathcal{V})$-SP-tree and so there is a branch B of T such that $A=\{B(2n)\}_{n\in\omega}\in\mathcal{U}$ and $\bigcup_{n\in\omega}B(2n+1)\in\mathcal{V}$. As in the proof of properness

$$\mathbb{T}^*=\bigcup_n Lev_{B(n)}\left(\mathbb{T}_{B\restriction(n+1)}\right)\in\mathbb{P}(\mathcal{V}).$$

$\mathbb{T}^*\subseteq\mathbb{T}^*_{B\restriction(2n+1)}$ for each n and $\mathbb{T}^*_{B\restriction(2n+1)}\Vdash_{\mathbb{P}(\mathcal{V})}$ "$B(2n)\in\dot{X}$". Hence $\mathbb{T}^*\Vdash_{\mathbb{P}(\mathcal{V})}$ "$A\subseteq\dot{X}$". □

At this stage it is already possible to obtain a model of set theory with a unique selective ultrafilter. Start with a model of $\diamondsuit_{\omega_2}$ and to select an arbitrary selective ultrafilter $\mathcal{V}$ in this model. Then construct a countable support iteration of partial orders $\mathbb{Q}_\xi$ of length ω_2 such that each ξ^{th} iterand is of the form $\mathbb{P}(\mathcal{U}_\xi)$ provided that the name $\mathcal{U}_\xi$ is guessed by the $\diamondsuit_{\omega_2}$ sequence and

$$\mathbf{1} \Vdash_{\mathbb{Q}_\xi} \text{“}\mathcal{V}_\xi \text{ is a P-point and } \mathcal{V} \not\leq_{\mathsf{RK}} \mathcal{U}_\xi\text{”}.$$

Each $\mathbb{Q}_\xi$ is proper and ω^ω-bounding. Hence, by Corollary 1 it follows that

$$\mathbf{1} \Vdash_{\mathbb{Q}_{\omega_2}} \text{“}\mathcal{V}_\xi \text{ is a not a P-point. ”}$$

Hence $\mathbf{1} \Vdash_{\mathbb{Q}_{\omega_2}}$ “ if $\mathcal{V}_\xi$ is a a P-point then $\mathcal{V} \leq_{\mathsf{RK}} \mathcal{U}_\xi$”.

Since selective ultrafilters are RK minimal it follows that $\mathcal{V}$ is the only possible selective ultrafilter in the generic model obtained by forcing with $\mathbb{Q}_{\omega_2}$.

In order to get a single P-point, and not just a single selective ultrafilter, an argument is needed for destroying P-points $\mathcal{V}$ such that $\mathcal{U} \leq_{\mathsf{RK}} \mathcal{V}$ while preserving $\mathcal{U}$ when $\mathcal{U}$ is selective. Constructing such a partial order and establishing its key properties with be the focus of the remainder of this lecture.

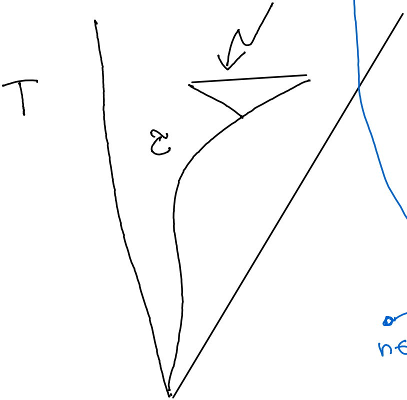
The following definition combines aspects of $\mathcal{U}$-P-trees and $(\mathcal{U}, \mathcal{V})$-SP-trees. Note that, unlike the case of $(\mathcal{U}, \mathcal{V})$-SP-trees, there is no difference between even and odd levels.

Definition 5

Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters and $h : \omega \to \omega$ a finite-to-one function witnessing that $\mathcal{U} \leq_{\mathsf{RK}} \mathcal{V}$. Define a tree T to be a $(\mathcal{U}, \mathcal{V}, h)$-SP-tree if for each $\tau \in T$ there are $A_\tau \in \mathcal{U}$ and $B_\tau \in \mathcal{V}$ such that

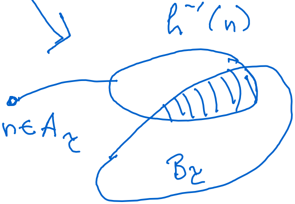
$$\mathbf{succ}_T(\tau) = \left\{ (n, h^{-1}\{n\} \cap B_\tau) \mid n \in A_\tau \right\}.$$

$$\mathrm{succ}_T(\tau) = \{\underbrace{(n, h^{-1}(n) \cap B_\tau)}\}_{n \in A_\tau}$$



$A_\tau \in \mathcal{U}$

$B_\tau \in \mathcal{V}$



Lemma 7

Let $\mathcal{U}$ be a selective ultrafilter and $\mathcal{V}$ a P-point such that $h : \omega \to \omega$ a finite-to-one function witnessing that $\mathcal{U} \leq_{\mathsf{RK}} \mathcal{V}$. Then for any $(\mathcal{U}, \mathcal{V}, h)$-SP-tree T there is a branch B such that letting $B(n) = (B_0(n), B_1(n))$

$$\{B_0(n) \mid n \in \omega\} \in \mathcal{U} \quad \& \quad \bigcup_n B_1(n) \in \mathcal{V}$$

Proof.

This uses ideas similar to those of the proof of Lemma 4. □

Definition 6

Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters and $h : \omega \to \omega$ a finite-to-one function witnessing that $\mathcal{V} \leq_{\mathrm{RK}} \mathcal{U}$. Define the partial order $\mathbb{P}(\mathcal{V}, \mathcal{U}, h)$ to consist of trees $\mathbb{T}$ such that

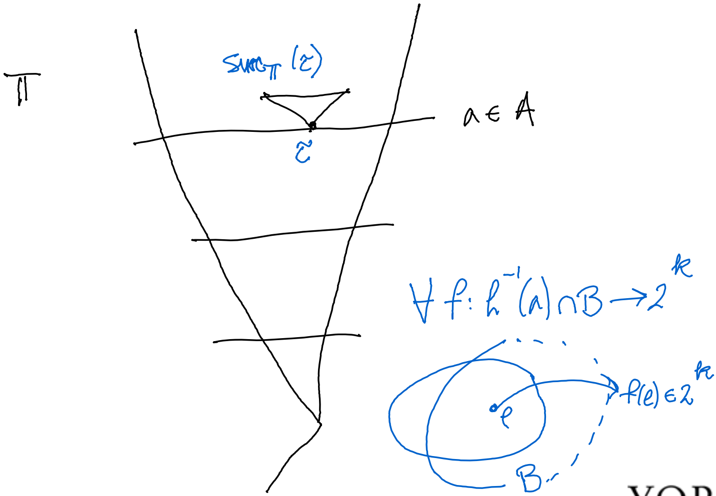
$$(\forall \tau \in \mathbb{T})\ \mathbf{succ}_{\mathbb{T}}(\tau) \subseteq \left(2^{|\tau|}\right)^{h^{-1}(|\tau|)} \tag{6}$$

and there are $A \in \mathcal{U}$ and $B \in \mathcal{V}$ such that for all $k \in \omega$

$$\begin{aligned}(\forall^{\infty} a \in A)(\forall t \in \mathrm{Lev}_a(\mathbb{T}))(\forall f : h^{-1}(a) \cap B \to 2^k)\\ (\exists g \in \mathbf{succ}_{\mathbb{T}}(t))(\forall j \in h^{-1}(a) \cap B)\ f(j) \subseteq g(j).\end{aligned} \tag{7}$$

Define C_G by letting $F_G(k) : h^{-1}(k) \to 2^k$ if for all $T \in G$ there is $t \in T$ such that $t(k) = F_G(k)$ and define

$$C_G(\ell, j) = \begin{cases} F_G(h(\ell))(\ell)(j) & \textit{if } j \in h(\ell) \\ 0 & \textit{otherwise.} \end{cases}$$


T
$\mathrm{SUCC}_T(\tau)$
τ
$a \in A$
$\forall f: h^{-1}(a) \cap B \to 2^k$
e
$f(e) \in 2^k$
B

Lemma 8

If $\mathcal{V}$ be a selective ultrafilter and $\mathcal{U}$ a P-point such that $h : \omega \to \omega$ a function witnessing that $\mathcal{V} \leq_{\mathsf{RK}} \mathcal{U}$ then $\mathbb{P}(\mathcal{V}, \mathcal{U}, h)$ is proper and ω^ω bounding.

Proof.

This is shown by argument similar to those that establish that $\mathbb{P}(\mathcal{U})$ is proper and ω^ω bounding.

$\square$

Lemma 9

Suppose that

- $\mathcal{U}$ *is a selective ultrafilter*
- $\mathcal{V}$ *is a P-point*
- h *a function witnessing that* $\mathcal{V} \leq_{\mathrm{RK}} \mathcal{U}$
- $\mathcal{V} \neq_{\mathrm{RK}} \mathcal{U}$
- $T \in \mathbb{P}(\mathcal{U}, \mathcal{V}, h)$
- $P : T \to 2$.

Then there is $T^* \subseteq T$ *such that* $T^* \in \mathbb{P}(\mathcal{U}, \mathcal{V}, h)$ *and there is* $W \in \mathcal{U}$ *and* $J \in 2$ *such that*

$$(\forall w \in W)(\forall t \in \mathrm{Lev}_{w+1} T^*)\ P(t) = J.$$

A sketch of the proof of this lemma uses the following:

Lemma 10

For arbitrary sets R and D if $R^D = P_0 \cup P_1$ then there is $d \in D$ and a partition $D \setminus \{d\} = D_0 \cup D_1$ such that for all $f : D_i \to R$ there is $f^* \in P_i$ such that $f \subseteq f^*$.

This lemma is used in the following context: $\dot{X}$ is a name for a subset of ω and

- $\tau \in \mathbb{T}$ and $A \in \mathcal{U}$ and $B \in \mathcal{V}$ witness that $\mathbb{T} \in \mathbb{P}(\mathcal{U}, \mathcal{V}, h)$
- $\mathbb{T}\langle \tau ^\frown f \rangle \Vdash_{\mathbb{P}(\mathcal{U},\mathcal{V},h)}$ "$\chi_{\dot{X}}(|\tau|) = J_f$" for $f \in \mathbf{succ}_{\mathbb{T}}(\tau)$.

Let $D = h^{-1}(|\tau|) \cap B$ and $R = 2^{|\tau|}$. Then for each $f \in R^D$ there is $g[f] \in \mathbf{succ}_{\mathbb{T}}(\tau)$ such that

$$(\forall j \in h^{-1}(|\tau|) \cap B) \;\; f(j) \subseteq g[f](j).$$

Let $R^D = P_0 \cup P_1$ be the partition defined by

$$P_i = \left\{ f \in R^D \;\middle|\; J_{g[f]} = i \right\}.$$

Lemma 10 then provides a partition

$$h^{-1}(|\tau|) \cap B = D = D_0 \cup D_1 \cup \{d\}$$

such that for all $f : D_i \to R$ there is $f^* \in P_i$ such that $f \subseteq f^*$. Letting D_i^τ denote D_i and d^τ denote d for a particular τ, an argument using Lemma 7 then yields $T^* \subseteq T$ such that $T^* \in \mathbb{P}(\mathcal{U}, \mathcal{V}, h)$ and $\bar{A} \in \mathcal{U}$ and $\bar{B} \in \mathcal{V}$ such that either:

- $D_0^\tau \supseteq h^{-1}(|\tau|) \cap \bar{B}$ for each $w \in W$ and $\tau \in \mathrm{Lev}_w(\mathbb{T}^*)$
- $D_1^\tau \supseteq h^{-1}(|\tau|) \cap \bar{B}$ for each $w \in W$ and $\tau \in \mathrm{Lev}_w(\mathbb{T}^*)$
- $\{d^\tau\} = h^{-1}(|\tau|) \cap \bar{B}$ for each $w \in W$ and $\tau \in \mathrm{Lev}_w(\mathbb{T}^*)$

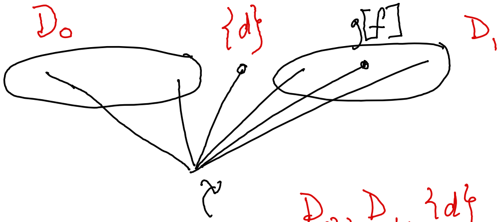
$R = 2^K$

for $f \in 2^K$ $\quad g[f] \in \mathrm{succ}_\pi(\tau)$

$\left\{ f \in 2^K \right\} g[k]$

$\pi\langle\tau\rangle^\frown f$ decides $|\tau| \in \dot{X}$

$P_0 \cup P_1$ is the resulting partition



$D_0, D_1, \{d\}$ from Lemma

If $J \in 2$ is such that one of the first two possibilities holds for J then

$$(\forall k \in W)(\forall \tau \in \mathsf{Lev}_k(\mathbb{T}^*))(\forall f : D^\tau_J \to 2^{|\tau|})$$
$$(\exists g \in \mathbf{succ}_{\mathbb{T}^*}(\tau))(\forall j \in D^\tau_J)\ f(j) \subset g(j) \quad (8)$$

and hence

$$(\forall k \in W)(\forall \tau \in \mathsf{Lev}_k(\mathbb{T}^*))(\forall f : h^{-1}(|\tau|) \cap \bar{B} \to 2^{|\tau|})$$
$$(\exists g \in \mathbf{succ}_{\mathbb{T}^*}(\tau))(\forall j \in D^\tau_J)\ f(j) \subset g(j) \quad (9)$$

and so $\mathbb{T}^* \Vdash_{\mathbb{P}(\mathcal{U},\mathcal{V},h)}$ "$(\forall w \in W)\ \chi_{\dot{X}}(w) = J$" as required.

The third possibility is ruled out by the hypothesis that $\mathcal{V} \neq_{\mathsf{RK}} \mathcal{U}$.

The immediate corollary now is the following.

Corollary 2

If $\mathcal{U}$ is a selective, $\mathcal{V}$ is a P-point and h witnesses that $\mathcal{V} \leq_{\mathsf{RK}} \mathcal{U}$ and $\mathbf{1} \Vdash_{\mathbb{P}(\mathcal{U},\mathcal{V},h)}$ "$\dot{X} \in \mathcal{U}^+$" then there is $\mathbb{T} \in \mathbb{P}(\mathcal{U},\mathcal{V},h)$ and $A \in \mathcal{U}$ such that

$$\mathbb{T} \Vdash_{\mathbb{P}(\mathcal{U},\mathcal{V},h))} \text{“}A \subseteq \dot{X}\text{”}.$$

There is only one final piece of the puzzle needed and it is provided the next lemma, whose proof is similar to the corresponding result for $\mathbb{P}(\mathcal{U})$.

Lemma 11

If $\mathcal{U}$ is a selective, $\mathcal{V}$ is a P-point and h witnesses that $\mathcal{V} \leq_{\mathsf{RK}} \mathcal{U}$ and $J \in 2$ and

$$\mathbf{1} \Vdash_{\mathbb{P}(\mathcal{V},\mathcal{U},h)} \text{“}\dot{X} \textit{ is almost-}J\textit{-homogeneous for } C_{\dot{G}}\text{”}$$

then there is $\mathbb{T} \in \mathbb{P}(\mathcal{V},\mathcal{U},h))$ and $E \in \mathcal{U}$ such that

$$\mathbb{T} \Vdash_{\mathbb{P}(\mathcal{V},\mathcal{U},h))} \text{“}E \cap \dot{X} = \varnothing\text{”}.$$

A countable support iteration, starting with a model of $\diamondsuit_{\omega_2}$ and a fixed selective ultrafilter $\mathcal{U}$, of partial orders $\mathbb{P}_\xi * \mathbb{Q}_\xi$ where

- $\mathbb{Q}_\xi = \mathbb{P}(\mathcal{V}_\xi)$ if $\diamondsuit_{\omega_2}$ at ξ guesses $\mathcal{V}_\xi$ and $\mathbf{1} \Vdash_{\mathbb{P}_\xi}$ “$\mathcal{U} \not\leq_{\mathsf{RK}} \mathcal{V}_\xi$”
- $\mathbb{Q}_\xi = \mathbb{P}(\mathcal{U}, \mathcal{V}_\xi, h)$ if $\diamondsuit_{\omega_2}$ at ξ guesses $\mathcal{V}_\xi$ and $\mathbf{1} \Vdash_{\mathbb{P}_\xi}$ “$\mathcal{U} \leq_{\mathsf{RK}} \mathcal{V}_\xi$”.

provides a model with a unique P-point.

It is not hard to modify this proof to get model of set theory with any specified number of RK-equivalence classes of P-points (but there is only homeomorphism class of P-points of character $\aleph_1$.)

Question 1

What RK structures are possible for the set of P-points?

Given that in the models discussed with some, but not many P-points, the P-points are all selective, one may ask whether it is possible to have P-points, but no selective ultrafilters.

Theorem 1 (Combining Kunen [2] and Dow [1])

In a model obtained by adding $\aleph_2$ random reals to a model of $V = L$ there are no selective ultrafilters, but there are P-points.

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