P-points and related ultrafilters — part II

Juris Steprāns

York University

Winter School — January 2024

This section will be present a method, due to Chodounský and Guzman [1], but following an alternate, streamlined proof due, in addition, to Verner. While very elegant and relatively straightforward, it seems to be very specific and has not leant itself to modifications, leaving open several questions.

Definition 1

$\mathbb{SI}$ denotes Silver forcing and consists of all partial functions $f : \omega \to 2$ such that $\mathrm{domain}(f)$ is co-infinite. The ordering on $\mathbb{SI}$ is inclusion. Given a set of ordinals X define

$$\mathbb{SI}^X = \left\{ F \in \mathbb{SI}^X \mid |\{\xi \in X \mid F(\xi) \neq \varnothing\}| \leq \aleph_0 \right\}$$

to be the countable support product of $\mathbb{SI}$ with the coordinate-wise inclusion ordering. For $F \in \mathbb{SI}^X$ define the support of F to be $\{\xi \in X \mid F(\xi) \neq \varnothing\}$. If $G \subseteq \mathbb{SI}^X$ is generic and $\xi \in X$ let the ξ^{th} Silver real be denoted by $\dot{S}_\xi = \bigcup_{F \in G} F(\xi)$.

YORK UNIVERSITÉ UNIVERSITY

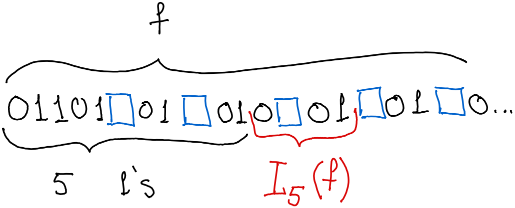
Definition 2

Given $f \in \mathbb{SI}$ such that $f^{-1}\{1\}$ is infinite and $j \in \omega$ define $I_j(f) = \left\{k \in \omega \;\middle|\; j \leq |f^{-1}\{1\} \cap k| < j+1\right\}$. For $J \in 2$ define

$$D_J(f) = \bigcup_{n\in\omega} I_{2n+J}(f)$$

and let this definition also apply to total functions $f : \omega \to 2$.

If $f^{-1}\{1\}$ is finite then some $I_j(f)$ will not be defined, but this will not play a role since the set of $f \in \mathbb{SI}$ such that $f^{-1}\{1\}$ is infinite is dense.


f
01101 01 010 01 01 0...
5 1's
$I_5(f)$

Key ideas of the proof

- The key observation is that for any $G \subseteq \mathbb{SI}^{\omega}$ generic over V, any ultrafilter $\mathcal{U}$ from V and $J \in 2$ there is some $U \in \mathcal{U}$ that is disjoint from any pseudo-intersection of the sets $\{D_J(\dot{S}_n)\}_{n\in\omega}$.
- In the model obtained by forcing with $\mathbb{SI}^{\omega_2}$ over a model of the Continuum Hypothesis, for every ultrafilter $\mathcal{U}$ in the ground model there is a countable $B \subseteq \omega_2$ from the ground model and some $J \in 2$ such that $D_J(\dot{S}_\xi) \in \mathcal{U}$ for each $\xi \in B$.
- One might hope that the final step would be to show that if $\mathcal{U}$ is an arbitrary ultrafilter in a generic extension by $G \subseteq \mathbb{SI}^{\omega_2}$ then there is a set $X \in [\omega_2]^{\aleph_1}$ such that $\mathcal{U} \cap V[G \cap \mathbb{SI}^X]$ is an ultrafilter and that $G \cap \mathbb{SI}^{\omega_2 \setminus X}$ is generic over $G \cap \mathbb{SI}^X$.
- However, this makes no sense since the forcing is a product rather than an iteration.

In the following lemma note that A is a set, not a name.

Lemma 1

Let $\mathbb{P}$ be a partial order such that $\mathbb{P} \times \mathbb{SI}^\omega$ is ω^ω-bounding. Suppose further that $\dot{\mathcal{U}}$ is a $\mathbb{P} \times \mathbb{SI}^\omega$-name for an ultrafilter such that for each $A \subseteq \omega$ the set $E_A(\dot{\mathcal{U}})$ defined to be

$$\left\{ p \in \mathbb{P} \;\middle|\; (p, \varnothing) \Vdash_{\mathbb{P}\times\mathbb{SI}^\omega} \text{“}A \in \dot{\mathcal{U}}\text{” or } (p, \varnothing) \Vdash_{\mathbb{P}\times\mathbb{SI}^\omega} \text{“}A \notin \dot{\mathcal{U}}\text{”} \right\} \tag{1}$$

is dense in $\mathbb{P}$. Then for each $J \in 2$

$$1 \Vdash_{\mathbb{P}\times\mathbb{SI}^\omega} \text{“}(\forall X \subseteq \omega)(\exists U \in \dot{\mathcal{U}}) \text{ either } (\exists n \in \omega)\ |X \setminus D_J(\dot{S}_n)| = \aleph_0 \text{ or } |X \cap U| < \aleph_0.\text{”} \tag{2}$$

Proof.

Let $\dot{Z}$ be a $\mathbb{P} \times \mathbb{SI}^\omega$ name such that

$$1 \Vdash_{\mathbb{P} \times \mathbb{SI}^\omega} \text{“}(\forall n \in \omega)\ \dot{Z} \subseteq^* D_J(\dot{S}_n)\text{”}.$$

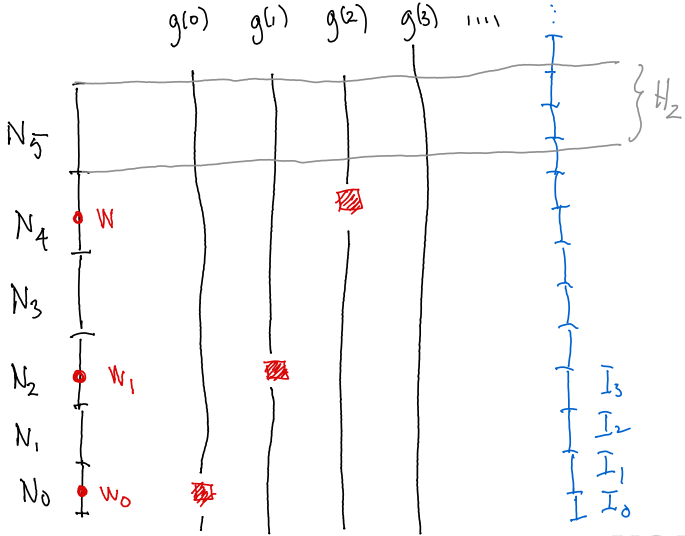
Since $\mathbb{P} \times \mathbb{SI}^\omega$ is ω^ω-bounding it is possible to partition ω into intervals $\{N_k\}_{k\in\omega}$ such that for some $(p, f) \in \mathbb{P} \times \mathbb{SI}^\omega$:

- $(p, f) \Vdash_{\mathbb{P} \times \mathbb{SI}^\omega}$ “$\dot{Z} \setminus \max(N_n) \subseteq D_J(\dot{S}_n)$” for each $n \in \omega$
- $N_n \setminus \text{domain}(f(i)) \neq \varnothing$ for each $i \leq n$.

It may be assumed that $\bar{U} = \bigcup_{n\in\omega} N_{2n+1} \in \mathcal{U}$. The goal now is to find a condition $(q, h) \leq (p, f)$ such that $(q, h) \Vdash_{\mathbb{P} \times \mathbb{SI}^\omega}$ “$|\dot{Z} \cap \bar{U}| < \aleph_0$”. As a first step, find $g \supseteq f$ such that for every $n \in \omega$ there is some $w_n \in N_{2n}$ such that

- $\bigcup_{\ell=0}^{2n+1} N_\ell \subseteq \text{domain}(g(n)) \cup \{w_n\}$
- $w_n \notin \text{domain}(g(n))$.

Now define $H_n = N_{2n+1} \cap D_J(g(n))$.


g(0)
g(1)
g(2)
g(3)
H_2
N_5
N_4
W
N_3
N_2
W_1
N_1
N_0
W_0
I_3
I_2
I_1
I_0

Proof continued.

Since the w_n are in the domains of different coordinates of g it is possible to extend g to $\bar{g}$ so that $\bar{g}(n)(w_n)$ takes on any value desired. In particular,

$$\text{if } \bar{g}(n)(w_n) = 0 \text{ then } \bar{g} \Vdash_{\mathbb{SI}^\omega} \text{“}N_{2n+1} \cap D_J(\dot{S}_n) = H_n\text{”}$$

while

$$\text{if } \bar{g}(n)(w_n) = 1 \text{ then } \bar{g} \Vdash_{\mathbb{SI}^\omega} \text{“}N_{2n+1} \setminus D_J(\dot{S}_n) = H_n\text{”}.$$

Define $H = \bigcup_{n\in\omega} H_n$, noting that $H \subseteq \bar{U}$. A modification to be made to $\bar{U}$ will depend on whether H is forced to belong to $\dot{\mathcal{U}}$ or not. Noting that H belongs to the ground model, let $q \leq p$ belong to $E_H(\dot{\mathcal{U}})$.

Proof continued.

If $(q, \varnothing) \Vdash_{\mathbb{P}\times\mathbb{SI}^\omega}$ "$H \notin \dot{\mathcal{U}}$" then let $U = \bar{U} \setminus H \in \mathcal{U}$. In this case let $h \supseteq g$ be such that $h(n)(w_n) = 0$ for every $n \in \omega$. We know that

$$h \Vdash_{\mathbb{SI}^\omega} \text{“}U \cap N_{2n+1} \cap D_J(\dot{S}_n) = U \cap H_n = \varnothing\text{”}.$$

Since $(q, h) \Vdash_{\mathbb{P}\times\mathbb{SI}^\omega}$ "$\dot{Z} \cap N_{2n+1} \subseteq \dot{Z} \setminus \max(N_n) \subseteq D_J(\dot{S}_n)$" it follows that $(q, h) \Vdash_{\mathbb{P}\times\mathbb{SI}^\omega}$ "$U \cap N_{2n+1} \cap \dot{Z} = \varnothing$".

Since $U \subseteq \bar{U} = \bigcup_n N_{2n+1}$ it follows that

$$(q, h) \Vdash_{\mathbb{P}\times\mathbb{SI}^\omega} \text{“}U \cap \dot{Z} \subseteq \bigcup_{n\in\omega} U \cap N_{2n+1} \cap \dot{Z} = \varnothing\text{”}.$$

YORK UNIVERSITÉ UNIVERSITY

Completion of proof.

On the other hand, if $(q, \varnothing) \Vdash_{\mathbb{P}\times\mathbb{SI}^\omega}$ "$H \in \dot{\mathcal{U}}$" then let $U = H$. In this case let $h \supseteq g$ be such that $h(n)(w_n) = 1$ for every $n \in \omega$.

It now also follows that $h \Vdash_{\mathbb{SI}^\omega}$ "$D_J(\dot{S}_n) \cap H_n = \varnothing$". Since

$$(q, h) \Vdash_{\mathbb{P}\times\mathbb{SI}^\omega} \text{“}\dot{Z} \cap N_{2n+1} \subseteq D_J(\dot{S}_n)\text{”}$$

it again follows that $(q, h) \Vdash_{\mathbb{P}\times\mathbb{SI}^\omega}$ "$H_n \cap \dot{Z} = \varnothing$" and, hence, that

$$(q, h) \Vdash_{\mathbb{P}\times\mathbb{SI}^\omega} \text{“}U \cap \dot{Z} = H \cap \dot{Z} = \varnothing\text{”}$$

as required. □

Lemma 2

If $2^{\aleph_0} = \aleph_1$ then for any $Y \subseteq \mathbb{SI}^{\omega_2}$ such that $|Y| = \aleph_2$ there is $\bar{Y} \in [Y]^{\aleph_2}$ such that for any $B \in [\bar{Y}]^{\aleph_0}$ there is $q \in \mathbb{SI}^{\omega_2}$ such that $q \supseteq \bigcup B$.

Proof.

Use a Δ-system argument and the hypothesis $2^{\aleph_0} = \aleph_1$. □

Lemma 3

If $2^{\aleph_0} = \aleph_1$ and $1 \Vdash_{\mathbb{SI}^{\omega_2}}$ "$\dot{\mathcal{U}}$ is an ultrafilter" then there is a dense set of $f \in \mathbb{SI}^{\omega_2}$ for which there is a countable $B \subseteq \omega_2$ and $J \in 2$ such that

- $f \Vdash_{\mathbb{SI}^{\omega_2}}$ "$D_J(\dot{S}_\beta) \in \dot{\mathcal{U}}$" *for all* $\beta \in B$
- *the set* $E_A(\dot{\mathcal{U}})$ *of (1) is dense in* $\mathbb{SI}^{\omega_2 \setminus B}$ *for each* $A \subseteq \omega$.

Recall that $E_A(\dot{\mathcal{U}})$ is defined to be

$$\left\{ p \in \mathbb{P} \;\middle|\; (p, \varnothing) \Vdash_{\mathbb{P} \times \mathbb{SI}^\omega} \text{“}A \in \dot{\mathcal{U}}\text{” or } (p, \varnothing) \Vdash_{\mathbb{P} \times \mathbb{SI}^\omega} \text{“}A \notin \dot{\mathcal{U}}\text{”} \right\} \tag{3}$$

and note that $\mathbb{P}$ is $\mathbb{SI}^{\omega_2 \setminus B}$ in this case.

YORK UNIVERSITÉ UNIVERSITY

Proof.

Let $g \in \mathbb{SI}^{\omega_2}$ be given. From Lemma 2 it follows that $\mathbb{SI}^{\omega_2}$ satisfies the $\aleph_2$-chain condition. Since $2^{\aleph_0} = \aleph_1$ it is possible to find $E \subseteq \mathbb{SI}^{\omega_2}$ such that $|E| = \aleph_1$ and such that $E_A(\dot{\mathcal{U}}) \cap E$ is dense in $\mathbb{SI}^{\omega_2}$ for each $A \subseteq \omega$. Let $X \in [\omega_2]^{\aleph_1}$ be so large that it contains the support of all $f \in E$.

For each $\xi \in \omega_2$ choose some $f_\xi \in \mathbb{SI}^{\omega_2}$ and $J_\xi \in 2$ such that $f_\xi \supseteq g$ and $f_\xi \Vdash_{\mathbb{SI}^{\omega_2}}$ "$D_{J_\xi}(\dot{S}_\xi) \in \dot{\mathcal{U}}$". Using Lemma 2 find $\Lambda \in [\omega_2]^{\aleph_2}$ such that $\bigcup_{\xi \in B} f_\xi \in \mathbb{SI}^{\omega_2}$ for each $B \in [\Lambda]^{\aleph_0}$. Let $J \in 2$ be such that if $\Lambda^* = \{\lambda \in \Lambda \mid J_\lambda = J\}$ then $|\Lambda^*| = \aleph_2$. Let $B \in [\Lambda^* \setminus X]^{\aleph_0}$ and define $f = \bigcup_{\xi \in B} f_\xi$.

□

YORK U UNIVERSITÉ UNIVERSITY

Theorem 1

If $2^{\aleph_0} = \aleph_1$ and $G \subseteq \mathbb{SI}^{\omega_2}$ is generic over V then $V[G]$ is a model of set theory with no P-points.

Proof.

If $\dot{\mathcal{U}}$ is a $\mathbb{SI}^{\omega_2}$ name for an ultrafilter then by the Lemma 3 there is some $f \in G$ and a countable $B \subseteq \omega_2$ and $J \in 2$ such that

- $f \Vdash_{\mathbb{SI}^{\omega_2}}$ "$D_J(\dot{S}_\beta) \in \dot{\mathcal{U}}$" for all $\beta \in B$
- the set $E_A(\dot{\mathcal{U}})$ of (1) is dense in $\mathbb{SI}^{\omega_2 \setminus B}$ for each $A \subseteq \omega$.

Then by Conclusion (2) of Lemma 1, with $\mathbb{P}$ taken to be $\mathbb{SI}^{\omega_2 \setminus B}$, it follows that

$$f \Vdash_{\mathbb{SI}^{\omega_2 \setminus B} \times \mathbb{SI}^B} \text{"}(\forall U \in \dot{\mathcal{U}})(\exists n \in \omega)\ |X \setminus D_J(\dot{S}_n)| = \aleph_0\text{"}$$

and, since $\mathbb{SI}^{\omega_2 \setminus B} \times \mathbb{SI}^B = \mathbb{SI}^{\omega_2}$, the result follows. □

Questions

Definition 3

A measure m satisfies **Property AP** *if for every pairwise disjoint sequence $\{A_n\}$ there is $A \subseteq \omega$ such that $A_n \subseteq^* A$ for all n and $m(A) = \sum_n m(A_n)$.*

An ultrafilter with Property AP is clearly a P-point. Mekler shows in [2] that it is consistent that there are no finitely additive measures with Property AP, thus strengthening Shelah's consistency that there are no P-points. Borodulin-Nadzieja, Cancino and Morawski that the existence of a measure with AP does not imply the existence of a P-point. However,

Question 1

Are there ultrafilters with Property AP in the Silver model?

The next part of this lecture will look at a model of Shelah [3] in which there are no Nowhere Dense (NWD) ultrafilters.

DEFINITION 4

Recall from the first lecture that if $\mathcal{I}$ is an ideal on X then $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter if for every $F : \omega \to X$ there is $A \in \mathcal{I}$ such that $F^{-1}(A) \in \mathcal{U}$.

Recall also from the first lecture that every P-point is a NWD ultrafilter, so this is a strengthening of having no P-points. So it begs the following question.

QUESTION 2

Are there NWD-ultrafilters in the Silver model?

Definition 5

A partial order $\mathbb{P}$ *will be said to have the* weak Sacks property *if for every* $g : \omega \to \omega$ *such that* $\lim_{n\to\infty} g(n) = \infty$ *and every* $p \Vdash_{\mathbb{P}}$ "$\dot{f} : \omega \to \omega$" *there are infinite* $A \subseteq \omega$, $F : A \to [\omega]^{<\aleph_0}$ *and* $q \leq p$ *such that:*

- $|F(n)| \leq g(n)$ *for all* $n \in A$
- $q \Vdash_{\mathbb{P}}$ "$(\forall n \in A)\ \dot{f}(n) \in F(n)$".

Lemma 4

If $\mathbb{P}$ *has the weak Sacks property and*

$$1 \Vdash_{\mathbb{P}} \text{“}\mathbb{Q} \text{ has the weak Sacks property”}$$

then $\mathbb{P} * \mathbb{Q}$ *has the weak Sacks property.*

The proof of Lemma 4 is technical and not sufficiently enlightening to be worth reproducing here.

The following can be proved using ideas familiar from the iteration theory of proper partial orders, in particular, the preservation of ω^ω-bounding forcing by countable support.

Lemma 5

If α is an ordinal and $\mathbb{P}$ is the countable support iteration of proper partial orders $\mathbb{P}_\xi$ for $\xi \in \alpha$ and if each $\mathbb{P}_\xi$ has the weak Sacks Property then so does $\mathbb{P}$.

Definition 6

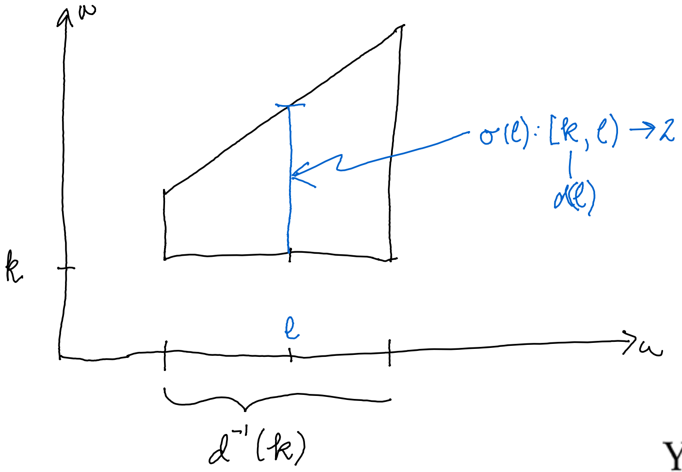
If $\mathcal{I}$ is an ideal define $\mathbb{P}(\mathcal{I})$ to consist of all functions σ defined on ω such that there is $d : \omega \to \omega$ satisfying that for each $n \in \omega$

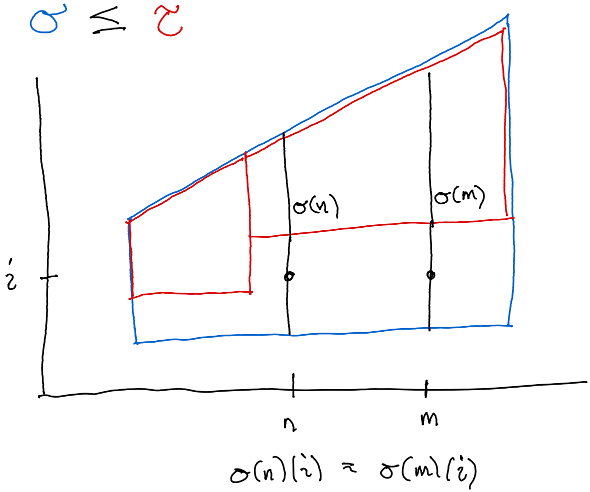
- $\sigma(n) : [d(n), n) \to 2$
- $d^{-1}(k) \in \mathcal{I}$ *for each* $k \in \omega$.

For $\sigma \in \mathbb{P}(\mathcal{I})$ let d_σ denote the function witnessing that $\sigma \in \mathbb{P}(\mathcal{I})$. For σ and τ in $\mathbb{P}(\mathcal{I})$ define $\sigma \leq \tau$ if

1. $\sigma(n) \supseteq \tau(n)$ *for all* n
2. *there is* $e : \omega \to \omega$ *such that* $e(n) \leq n$ *and* $d_\sigma(n) = e \circ d_\tau(n)$ *for all* n
3. *if* $d_\tau(n) = d_\tau(m)$ *and* $d_\sigma(n) \leq i < d_\tau(n)$ *then* $\sigma(n)(i) = \sigma(m)(i)$.

If $G \subseteq \mathbb{P}(\mathcal{I})$ is generic define $s_n^G = \bigcup_{\sigma \in G} \sigma(n)$.


ω
k
σ(ℓ) : [k, ℓ) → 2
d(ℓ)
ℓ
ω
d⁻¹(k)


σ ≤ τ
σ(n)
σ(m)
i
n
m
σ(n)(i) ≈ σ(m)(i)

Lemma 6

$\mathbb{P}(\mathcal{I})$ *is a partial order.*

Proof.

It only needs to be verified that $\mathbb{P}(\mathcal{I})$ is transitive, so suppose that $\sigma \leq \tau \leq \theta$. Then there are e and $\bar{e}$ such that $d_\tau = e \circ d_\theta$ and $d_\sigma = \bar{e} \circ d_\tau$. Then $\bar{e} \circ e$ satisfies Condition (2) required for $\sigma \leq \theta$.

To see that Condition (3) holds suppose that $d_\theta(n) = d_\theta(m)$ and $d_\sigma(n) \leq i < d_\theta(n)$. Then note that $d_\sigma(n) \leq d_\tau(n) \leq d_\tau(n)$ and so either $d_\sigma(n) \leq i < d_\tau(n)$ or $d_\tau(n) \leq i < d_\theta(n)$. If the first case holds note that $d_\tau(n) = e(d_\theta(n)) = e(d_\theta(m)) = d_\tau(m)$ and so $\sigma(n)(i) = \sigma(m)(i)$ because $\sigma \leq \tau$. In the second case use that $\tau \leq \theta$. □

Definition 7

For σ and τ in $\mathbb{P}(\mathcal{I})$ define $\sigma \leq_n \tau$ if $d_\sigma^{-1}\{i\} = d_\tau^{-1}\{i\}$ for all $i \in n$ in the range of d_τ.

It is easy to see that the set of $\sigma \in \mathbb{P}(\mathcal{I})$ such that the range of d_σ is ω is dense. While this is not essential, it is worth keeping in mind since it simplifies, somewhat, the following technical definition.

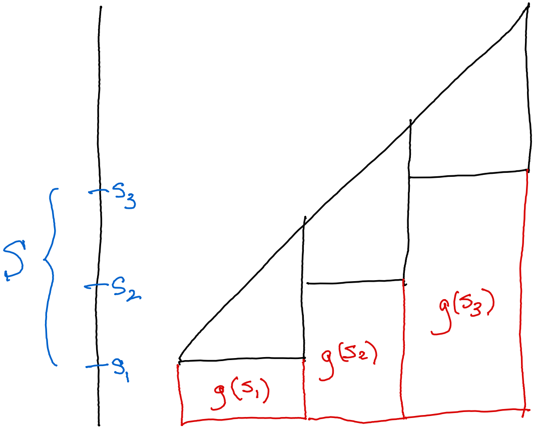
Definition 8

For $\sigma \in \mathbb{P}(\mathcal{I})$, $S \in [\omega]^{<\aleph_0}$ *and* $g \in \prod_{j\in S} 2^j$ *define* $\sigma[g]$ *by*

$$\sigma[g](\ell) = \begin{cases} \sigma(\ell) & \text{if } d_\sigma(\ell) \notin S \\ g(d_\sigma(\ell)) \cup \sigma(\ell) & \text{if } d_\sigma(\ell) \in S. \end{cases}$$

If $S \subseteq \omega$ *and* $\vec{g}$ *is defined on* $\bigcup_{n\in S} d_\sigma^{-1}(n)$ *and* $\vec{g}(i) \in 2^n$ *whenever* $d_\sigma(i) = n \in S$ *then let* $\sigma[\vec{g}]$ *be defined by*

$$\sigma[\vec{g}](\ell) = \begin{cases} \sigma(\ell) & \text{if } d_\sigma(\ell) \notin S \\ \vec{g}(\ell) \cup \sigma(\ell) & \text{if } d_\sigma(\ell) \in S. \end{cases}$$



S
s3
s2
s1
g(s1)
g(s2)
g(s3)

While the following lemma concluding that $\sigma[g] \leq \sigma$ is immediate, it may not be the case that $\sigma[\vec{g}] \leq \sigma$ because Condition (3) of Definition 6 may fail. When it is necessary to use S in the context of Definition 8 it will usually be a singleton, but there is a crucial point, Corollary 1, at which an infinite S will be needed.

Lemma 7

If $\sigma \in \mathbb{P}(\mathcal{I})$ and $n \in \omega$ and $g \in \prod_{j \in n} 2^j$ then $\sigma[g] \leq \sigma$.

Lemma 8

If $\tau \in \mathbb{P}(\mathcal{I})$ and $n \in \omega$ are such that there are

- $\vec{g}_i : d_\tau^{-1}(i) \to 2^{d_\tau(i)}$ *for each* $i \in n$
- *a dense set* $D \subseteq \mathbb{P}(\mathcal{I})$

then there are $\sigma \leq_n \tau$ *and* $\vec{g}_n : d_\tau^{-1}(n) \to 2^{d_\tau(n)}$ *and* $W \subseteq D$ *such that* $|W| < 2^{n^2}$ *and such that for each* $S \subseteq n$ *if* $\sigma_S = \sigma\left[\bigcup_{j\in S}\vec{g}_j\right]$ *then* $\sigma_S[\vec{g}_n] \leq \sigma_S$ *and* W *is predense below each* σ_S.

Note that it is **not** being claimed that $\sigma_S \leq \sigma$.

Proof Jump to Lemma 9 Jump to Theorem.

Let $\{S_j\}_{j\in 2^n}$ enumerate all the subsets of n and let $\{h^j_k\}_{k=0}^{m(j)}$ enumerate $\prod_{\ell\in n\setminus S_j} 2^\ell$ and note that it can assumed that $\sum_{\ell\in 2^n} m(\ell) < 2^{n^2}$. Construct inductively conditions τ^ℓ_k for $\ell \in 2^n$ such that:

- $\tau^0_0 = \tau$
- $\tau^{\ell+1}_0 = \tau^\ell_{m(\ell)}$
- $\tau^\ell_{k+1}[h^\ell_k \cup \bigcup_{m\in S_\ell} \vec{g}_m] \leq \tau^\ell_k[\bigcup_{m\in S_\ell} \vec{g}_m]$ and, hence, $\tau^\ell_{k+1} \leq \tau^\ell_k$
- $d^{-1}_{\tau^\ell_k}\{j\} = d^{-1}_\tau\{j\}$ if $j \in n$ is in the range of d_τ
- $\tau^\ell_{k+1}[h_k \cup \bigcup_{m\in S_\ell} \vec{g}_m] \in D$.

YORK UNIVERSITÉ UNIVERSITY

Continuation of proof of Lemma 8.

This is a standard argument, at least for each ℓ individually, but it is worth repeating in this context. Given τ^ℓ_k let $\bar\tau^\ell_k \leq \tau^\ell_k[h^\ell_k \cup \bigcup_{m\in S_\ell} \vec{g}_m]$ be such that $\bar\tau^\ell_k \in D$. Then let τ^ℓ_{k+1} be defined by

$$\tau^\ell_{k+1}(j) = \begin{cases} \bar\tau^\ell_k(j) & \text{if } d_\tau(j) \notin n \\ \tau(j) & \text{otherwise} \end{cases}$$

and note that $\bar\tau^\ell_k(j) \restriction [d_\tau(j), j) = \tau(j)$ if $d_\tau(j) \in n$. Keep in mind that it is not being claimed that $\tau^\ell_k[h^\ell_k \cup \bigcup_{m\in S_\ell} \vec{g}_m] \leq \tau^\ell_k$. However, since $\tau^\ell_k[h^\ell_k \cup \bigcup_{m\in S_\ell} \vec{g}_m]$ is, nevertheless, an element of $\mathbb{P}(\mathcal{I})$ there is no problem in finding $\bar\tau^\ell_k$.

Continuation of proof of Lemma 8.

Then extend $\tau^{2^n-1}_{m(2^n-1)}$ to τ^* so that if $d_\tau(j) \notin n$ and $d_{\tau^*}\{j\} \in n$ then $d_{\tau^*}\{j\} = 0$. If follows that $d^{-1}_{\tau^*}\{j\} = d^{-1}_{\tau}\{j\}$ if $0 < j < n$ and j is in the range of d_τ, but some more work is needed to get a condition σ such that $\sigma \leq_n \tau$.

To this end, let e be such that $d_{\tau^*} = e \circ d_\tau$ and define $\tilde{e}$ by

$$\tilde{e}(j) = \begin{cases} n & \text{if } e(j) = 0 \ \& \ d_\tau(j) \neq 0 \\ j & \text{otherwise.} \end{cases}$$

It is easy to verify that $\tilde{e}(j) \leq j$ for all j. Moreover, if $d_\sigma = \tilde{e} \circ d_{\tau^*}$ then $d^{-1}_\sigma\{j\} = d^{-1}_{\tau^*}\{j\} \in \mathcal{I}$ if $j \neq n$ and

$$d^{-1}_\sigma\{n\} \subseteq d^{-1}_{\tau^*}(0) \cup d^{-1}_{\tau^*}\{n\} \in \mathcal{I}.$$

Continuation of proof of Lemma 8.

Therefore, if σ is defined by setting $d_\sigma = \tilde{e} \circ d_{\tau^*}$ and letting

$$\sigma(j) = \begin{cases} \tau^*(j) & \text{if } \tilde{e}(j) \neq n \\ \tau^*(j) \restriction [n, j) & \text{if } \tilde{e}(j) = n \end{cases}$$

it follows that $\sigma \in \mathbb{P}(\mathcal{I})$.

Let $\vec{g}_n(i) = \tau^*(i) \restriction n$ for each $i \in d_{\tau^*}^{-1}\{0\} \setminus d_\tau^{-1}\{0\}$. It follows that $\sigma[\vec{g}_n] = \tau^*$ and, furthermore, $\sigma \leq_n \tau$.

Continuation of proof of Lemma 8.

Moreover, if $S \subseteq n$ and $S = S_\ell$ then $\{\tau_k^\ell\}_{k=0}^{m(\ell)}$ is predense below $\tau^\ell_{m(\ell)}\left[\bigcup_{m\in S_\ell}\vec{g}_m\right]$ and, since

$$\sigma\left[\vec{g}_n \cup \bigcup_{m\in S_\ell}\vec{g}_m\right] = \sigma\left[\bigcup_{m\in S_\ell}\vec{g}_m\right][\vec{g}_n] = \tau^*\left[\bigcup_{m\in S_\ell}\vec{g}_m\right] \leq \tau^\ell_{m(\ell)}\left[\bigcup_{m\in S_\ell}\vec{g}_m\right] \quad (4)$$

it follows that the conclusion holds with
$W = \left\{\tau_k^\ell \mid \ell \in 2^n \;\&\; k \in m(\ell)\right\}$. □

Lemma 9

If $\{\sigma_n\}_{n\in\omega} \subseteq \mathbb{P}(\mathcal{I})$ *and* $\sigma_{n+1} \leq_n \sigma_n$ *for each* $n \in \omega$ *then* $\sigma = \bigcup_{n\in\omega} \sigma_n \in \mathbb{P}(\mathcal{I})$.

Proof.

Note that $\{d^{-1}_{\sigma_{n+1}}\{n\}\}_{n\in\omega}$ are pairwise disjoint and, since

$$n \subseteq \bigcup_{k\in n} d^{-1}_{\sigma_{n+1}}\{k\}$$

it follows that $\bigcup_{n\in\omega} d^{-1}_{\sigma_{n+1}}\{n\} = \omega$. Hence, if d_σ is defined by $d_\sigma(j) = d_{\sigma_{n+1}}\{n\}$ if and only $d_{\sigma_{n+1}}(j) = d_{\sigma_{n+1}}(n)$ then d_σ witnesses that Definition 6 is satisfied by σ. □

YORK UNIVERSITÉ UNIVERSITY

Lemma 10

If $\tau \in \mathbb{P}(\mathcal{I})$ and $D_n \subseteq \mathbb{P}(\mathcal{I})$ are dense for $n \in \omega$ then there is $\sigma \leq \tau$ such that for each $j > 0$ there is $\vec{g}_j : d_\sigma^{-1}\{j\} \to 2^j$ and a finite set $W_j \subseteq D_j$ such that $|W_j| < 2^{j^2}$ and for any $S \subseteq j$ if $\sigma_S = \sigma\left[\bigcup_{i\in S} \vec{g}_i\right]$ then $\sigma_S[\vec{g}_j] \leq \sigma_S$ and W_j is predense below each σ_S.

Proof.

This follows directly from Lemma 8 and Lemma 9

□

Definition 9

Define an ideal $\mathcal{I}$ *to be* very tall *if for every partition* $\omega = \bigcup_{n\in\omega} A_n$ *such that* $A_n \in \mathcal{I}$ *for each* n *there is an infinite* $Z \subseteq \omega$ *such that* $\bigcup_{n\in Z} A_n \in \mathcal{I}$.

Corollary 1

If $\mathcal{I}$ *is very tall then* $\mathbb{P}(\mathcal{I})$ *has the weak Sacks property.*

PROOF.

Suppose that $\tau \Vdash_{\mathbb{P}(\mathcal{U})}$ "$\dot{f} : \omega \to \omega$" and $\lim_{n\to\infty} g(n) = \infty$. Let k_n be so large that $g(k_n) > 2^{n^2}$ and let $\dot{F}(n) = \dot{f}(k_n)$ and let D_n be the dense set of conditions deciding the value of $\dot{F}(n)$. Let $\sigma^* \leq \tau$ satisfy the conclusion of Lemma 10 for $\{D_j\}_{j\in\omega}$ and let $\vec{g}_j$ be the functions guaranteed by that lemma. Since the $d_{\sigma^*}^{-1}\{j\}$ are pairwise disjoint and $\mathcal{I}$ is very tall, it is possible to find an infinite $Z \subseteq \omega$ such that

$$\bigcup_{j\in Z} d_{\sigma^*}^{-1}\{j\} \in \mathcal{I}.$$

Let $\vec{g} = \bigcup_{i\in Z} \vec{g}_i$ and define $\sigma = \sigma^*[\vec{g}]$ and $X = \{k_n\}_{n\in Z}$. Then

$$\sigma = \sigma^*[\vec{g}] \Vdash_{\mathbb{P}(\mathcal{U})} \text{“}\dot{f}(k_n) = \dot{F}(n) \in W_n\text{”}$$

for each $k_n \in X$ and some set W_n such that $|W_n| \leq 2^{n^2} < g(k_n)$ as required. □

Corollary 2

$\mathbb{P}(\mathcal{I})$ is proper.

Proof.

Let $\mathfrak{M}$ be an elementary submodel of $\mathsf{H}(\kappa)$ such that $\mathcal{I} \in \mathfrak{M}$ and let $\tau \in \mathfrak{M} \cap \mathbb{P}(\mathcal{I})$. Let $\{D_n\}_{n\in\omega}$ enumerate all the dense subsets of $\mathbb{P}(\mathcal{I})$ in $\mathfrak{M}$ and assume that $D_{n+1} \subseteq D_n$. Then use Lemma 10 as in Corollary 1 to find $\sigma \leq \tau$ such that for infinitely many j there is a finite set $W_j \subseteq D_j$ such that and W_j is predense below each σ. It has to be noted that, since the argument of Corollary 1 can be carried out in $\mathfrak{M}$, it follows that the W_j can be assumed to be subsets of $\mathfrak{M}$. Therefore σ is $\mathfrak{M}$ generic.

□

Lemma 11

A set $Z \subseteq 2^\omega$ *is nowhere dense if and only if there are* $t_n : D_n \to 2$ *such that:*

- D_n *is a finite interval in* ω
- $\max(D_n) < \min(D_{n+1})$
- *if* $z \in Z$ *then* $t_n \not\subseteq z$ *for all* $n \in \omega$.

Proof.

Well known and easy.

□

Lemma 12

If $\mathbb{P}$ has the weak Sacks property and $1 \Vdash_{\mathbb{P}}$ "$\dot{Z} \subseteq 2^{\omega}$ is nowhere dense" then there is $p \in \mathbb{P}$, an infinite set $X \subseteq \omega$ and $t_n : [n, D_n) \to 2$ such that:

$$p \Vdash_{\mathbb{P}} \text{"}(\forall z \in \dot{Z})(\forall n \in X)\ t_n \not\subseteq z\text{"}.$$

Proof Jump to Theorem.

Using Lemma 11 let $\dot{s}_n$ be $\mathbb{P}$-names such that

- $1 \Vdash_{\mathbb{P}}$ "$(\forall n \in \omega)\ \dot{s}_n : \dot{E}_n \to 2$"
- $1 \Vdash_{\mathbb{P}}$ "$(\forall n \in \omega)\ \max(\dot{E}_n) < \min(\dot{E}_{n+1})$"
- $1 \Vdash_{\mathbb{P}}$ "$(\forall z \in \dot{Z})(\forall n \in \omega)\ \dot{s}_n \not\subseteq z$".

Continuation of proof of Lemma 11.

Let $\dot{w}_n = \{\dot{s}_j\}_{j=n}^{n+n}$ and use the weak Sacks property of $\mathbb{P}$ to find $p \in \mathbb{P}$, and infinite $X \subseteq \omega$ and W_n for $n \in X$ such that $p \Vdash_{\mathbb{P}}$ "$(\forall n \in \omega)\ \dot{w}_n \in W_n$" and $|W_n| = n$.

An easy inductive argument shows that for each $n \in X$ there is some D_n and $t_n : [n, D_n) \to 2$ such that for each $\{v_j\}_{j=n}^{n+n} \in W_n$ there is some j such that $n \leq j < n+n$ and $v_j \subseteq t_n$. It follows that $\{t_n\}_{n \in X}$ provides the desired sequence. □

YORK UNIVERSITÉ UNIVERSITY

Theorem 2

If $\mathcal{I}$ is very tall and $\dot{F}$ is any $\mathbb{P}(\mathcal{I})$-name such that

$$\varnothing \Vdash_{\mathbb{P}(\mathcal{I})} \text{“}\dot{F} : \omega \to 2^\omega \ \ \& \ \ (\forall n \in \omega)\ s_n^G \subseteq \dot{F}(n)\text{”}$$

and $1 \Vdash_{\mathbb{P}(\mathcal{I})}$ *“$\mathbb{Q}$ has the weak Sacks property” then for any (τ, p) such that*

$$(\tau, p) \Vdash_{\mathbb{P}(\mathcal{I})*\mathbb{Q}} \text{“}\dot{Z} \subseteq 2^\omega \text{ is nowhere dense.”}$$

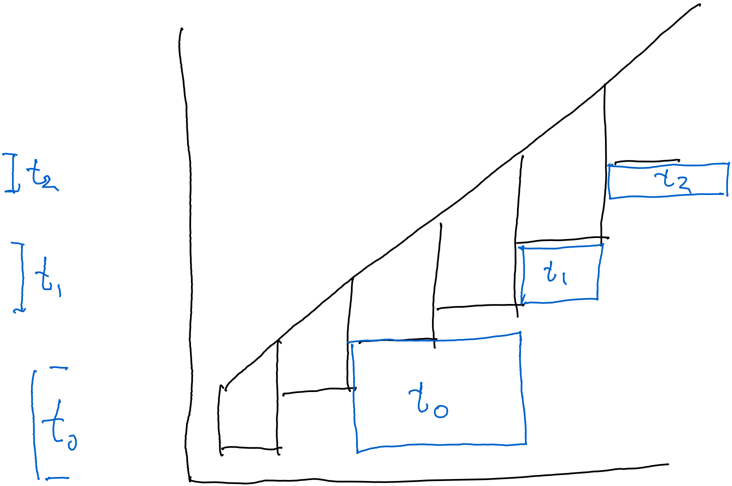
Then there are $(\sigma, q) \leq (\tau, p)$ and $U \in \mathcal{I}^*$ such that $(\sigma, q) \Vdash_{\mathbb{P}(\mathcal{I})*\mathbb{Q}}$ “$\dot{F}(U) \cap \dot{Z} = \varnothing$”.

Proof.

Use Lemma 4, Lemma 12 and Corollary 1 to find $(\tau^*, q) \leq (\tau, p)$, $t_n : [n, D_n) \to 2$ and an infinite set $X \subseteq \omega$ such that

$$(\tau^*, q) \Vdash_{\mathbb{P}(\mathcal{I})*\mathbb{Q}} \text{“}(\forall z \in \dot{Z})(\forall n \in X)\ t_n \not\subseteq z\text{”}.$$

Let $\{x_n\}_{n\in\omega}$ enumerate X in increasing order and, by choosing an infinite subset of X, it can be assumed that $D_{x_i} < x_{i+1}$.



t_2
t_1
t_0
t_2
t_1
t_0

Continuation of proof.

Then define the function e by

$$e(n) = \begin{cases} n & \text{if } n < D_{x_0} \\ x_i & \text{if } D_{x_i} \leq n < D_{x_{i+1}} \end{cases}$$

and define σ by

$$\sigma(n) = \begin{cases} \tau^*(n) & \text{if } n < D_{x_0} \\ t_{x_i} \cup \tau^*(n) & \text{if } D_{x_i} \leq n < D_{x_{i+1}}. \end{cases}$$

A key point to notice here is that Condition 3 of Definition 6 is satisfied and so $\sigma \leq \tau^*$. Let $U = \omega \setminus \bigcup_{j \in D_{x_0}} d_{\tau^*}^{-1}\{j\}$ and note that $U \in \mathcal{I}^*$ and

$$(\sigma, q) \Vdash_{\mathbb{P}(\mathcal{I}) * \mathbb{Q}} \text{``}(\forall n \in U)(\exists m \in X)\ t_m \subseteq \dot{F}(n)\text{''}$$

as required. □

Corollary 3

It is consistent that there are no nowhere dense ultrafilters; indeed no nowhere dense very tall filters .

Proof.

Let V be a model of set theory such that $\diamondsuit_{\omega_2}$ holds and this is witnessed by $\{D_\alpha\}_{\alpha\in\omega_2}$. Construct a countable support iteration $\{\mathbb{P}_\alpha\}_{\alpha\in\omega_2}$ such that for each $\alpha \in \omega_2$ if D_α is a $\mathbb{P}_\alpha$ name such that $1 \Vdash_{\mathbb{P}_\alpha}$ "D^*_α is an ultrafilter" then $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{P}(D_\alpha)$. Since the iteration is proper it follows that if $\dot{\mathcal{U}}$ is a $\mathbb{P}_{\omega_2}$ name for an ultrafilter then there is $\alpha \in \omega_2$ such that for any generic $G \subseteq \mathbb{P}_{\omega_2}$ the interpretation of D^*_α in $V[G \cap \mathbb{P}_\alpha]$ is equal to the interpretation of $\dot{\mathcal{U}}$ in $V[G \cap \mathbb{P}_\alpha]$.

Continuation of proof.

Since the ideals dual to an ultrafilter are easily seen to be very tall, it follows from Lemma 5 that in the model $V[G \cap \mathbb{P}_\alpha]$

$$1 \Vdash_{\mathbb{P}(D_\alpha)} \text{“}\mathbb{P}_{\omega_2}/\mathbb{P}_{\alpha+1} \text{ has the weak Sacks property.”}$$

It follows from Theorem 2 and Lemma 12 that there is a $\mathbb{P}_\alpha$-name for function $F : \omega \to 2^\omega$ such that if $F(U)$ is nowhere dense then $U \in \mathcal{U}^*$.

□

This proof actually shows that there are no very tall, NWD ideals.

Question

Shelah has shown that ultrafilters with some properties, such a Property M, are NWD-utrafilters and so do exist in the model of Corollary 3. On the other hand, letting $\mathcal{B}$ be the ideal on $\bigoplus_{n\in\omega}\{n\} \times n$ defined by

$$\mathcal{B} = \{X \subseteq \bigoplus_{n\in\omega}\{n\} \times n \mid (\exists k \in \omega)(\forall m \in \omega)\ |X \cap \{m\} \times m| < k\}$$

it is consistent with set theory that there is a $\mathcal{B}$-ultrafilter, but there are no nowhere dense ultrafilters.

Question 3

Determine for which other ideals $\mathcal{I}$ there are, or are not, $\mathcal{I}$-ultrafilters in the model of Corollary 3?

($\mathcal{U}$ has Property M if for all $\delta > 0$ and $\{X_n\}_n$ such that $\lambda(X_n) > \delta$ there is $A \in \mathcal{U}$ such that $\lambda(\bigcap_n X_n) > 0$.)

David Chodounský and Osvaldo Guzmán.
There are no P-points in Silver extensions.
Israel J. Math., 232(2):759–773, 2019.

Alan H. Mekler.
Finitely additive measures on N and the additive property.
Proc. Amer. Math. Soc., 92(3):439–444, 1984.

Saharon Shelah.
There may be no nowhere dense ultrafilter.
In *Logic Colloquium '95 (Haifa)*, volume 11 of *Lecture Notes Logic*, pages 305–324. Springer, Berlin, 1998.