Determinacy, Large Cardinals, and Inner Models

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Abstractly: Gödel's incompleteness theorems.

→ These are statements that are independent from ZFC

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"How many reals are there?"

Let us focus on the Continuum Problem:

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Actually, the picture is more complicated...

Determinacy Axioms: Games in set theory

Fix a set $A \subseteq {}^{\mathbb{N}}\mathbb{N}$.

infinite sequences of natural numbers

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Player I wins iff
$(n_0, n_1, \dots) \in A$.
O/w, Player II wins.

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A function $\sigma : \mathbb{N}^{<\mathbb{N}} \longrightarrow \mathbb{N}$ is a winning strategy for I in $G(A)$ iff

I	$\sigma(\emptyset)$		$\sigma(\sigma(\emptyset), n_1)$		$\dots$
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Def: The set A is determined iff one of the players has a winning strategy.

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Def: The set A is determined iff one of the players has a winning strategy.

The Axiom of Determinacy says:
Every set $A \subseteq {}^{\mathbb{N}}\mathbb{N}$ is determined.

What is determinacy good for?

Theorem (Mycielski, Swierczkowski, Mazur, Davis, 60's)

If all sets of reals are determined, then all sets of reals

- *are Lebesgue measurable,*
- *have the Baire property, and*
- *have the perfect set property.*

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If all sets of reals are determined and X is a zero-dimensional homogeneous space that is not locally compact, then X is strongly homogeneous.

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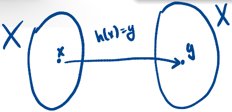

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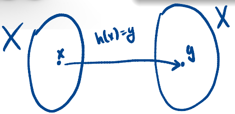

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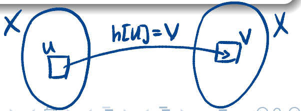

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How about other axioms?

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How far are these axioms from ZFC? "Steel's Program"

Consider hierarchies of these axioms and compare their strength.

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Determinacy

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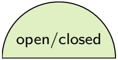
Forcing Axioms

Which games are determined?



Gale-Stewart (1953), ZFC

Theorem (Gale-Stewart, 1953)

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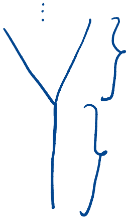
basic open sets:



arbitrary extensions

finite sequence of natural numbers

Theorem (Gale-Stewart, 1953)

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Proof.

Claim

Let $s \in {}^{2n}\mathbb{N}$. *If I does not have a winning strategy in the game starting with* s, *then for any* $i \in \mathbb{N}$, *there is a* $j \in \mathbb{N}$ *such that I does not have a winning strategy in the game starting with* $s^\frown(i, j)$.

Theorem (Gale-Stewart, 1953)

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Proof.

Suppose not and let i be a counterexample. For any $j \in \mathbb{N}$ let σ_j be a winning strategy for I in the game starting with $s^\frown(i, j)$.

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This yields a winning strategy for I in the game starting with s: Play i. If II responds with some j, continue playing according to σ_j. $\square$

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Suppose I does not have a winning strategy.

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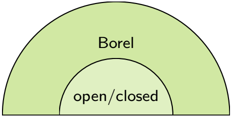
This τ **is a winning strategy for II**: Suppose not and let x be according to τ such that $x \in A$. As A is open, there is some basic open set $\mathcal{O}(x \upharpoonright 2n) \subseteq A$. But then any strategy for I in the game starting with $x \upharpoonright 2n$ is winning, contradicting the definition of τ. $\square$

Which games are determined?



Martin (1975), ZFC

Gale-Stewart (1953), ZFC

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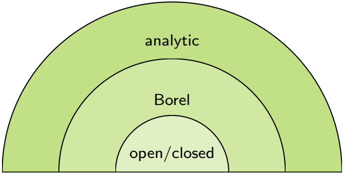

Martin (1970), measurable cardinal

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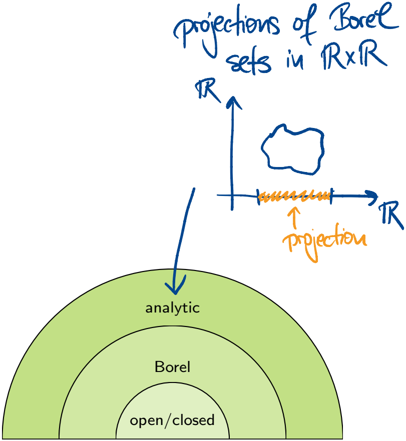

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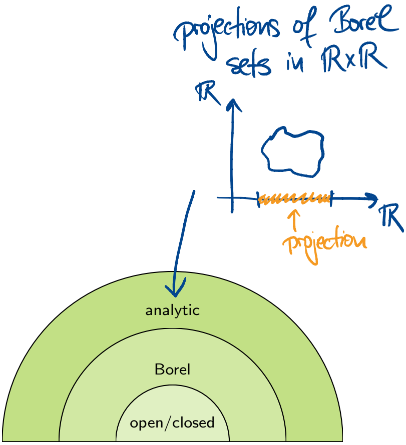

<u>Theorem (Martin)</u>: Suppose that $x^\#$ exists for every real x. Then every analytic set $B \subseteq {}^\omega\omega$ is determined.

Martin (1970), measurable cardinal

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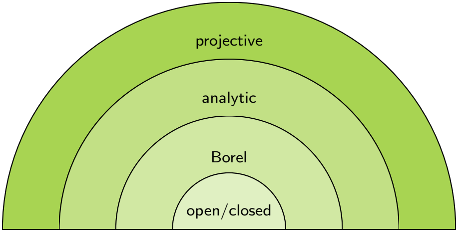
Gale-Stewart (1953), ZFC

Which games are determined?



Martin-Steel (1985), Woodin cardinals and a measurable cardinal

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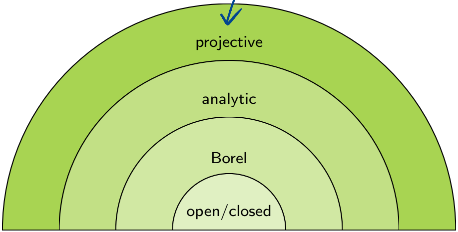
Which games are determined?

Iteratively build projections & complements



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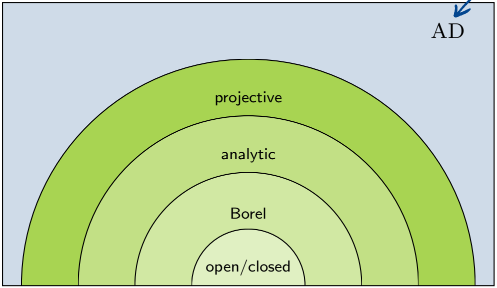
Which games are determined?

all games are determined



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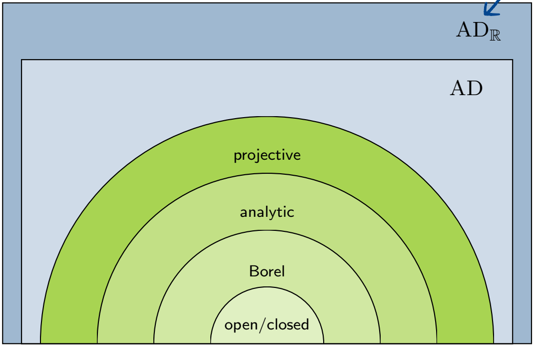
Which games are determined?

all games on $\mathbb{R}$ are determined

$$\begin{array}{c|cccc} \mathrm{I} & x_0 \in \mathbb{R} & & x_2 \in \mathbb{R} & \dots \\ \hline \mathrm{II} & & x_1 \in \mathbb{R} & & x_3 \in \mathbb{R} \dots \end{array}$$



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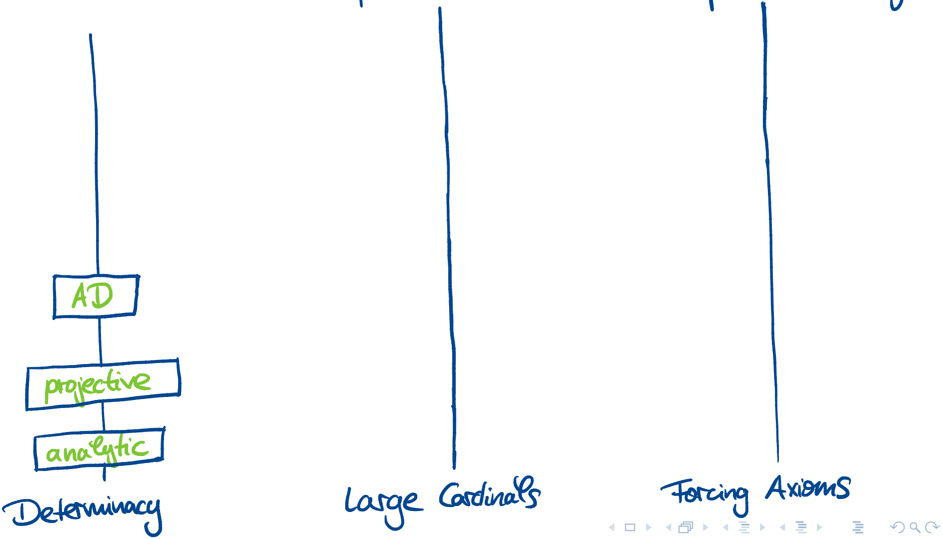
How far are these axioms from ZFC? "Steel's Program"

Consider hierarchies of these axioms and compare their strength.



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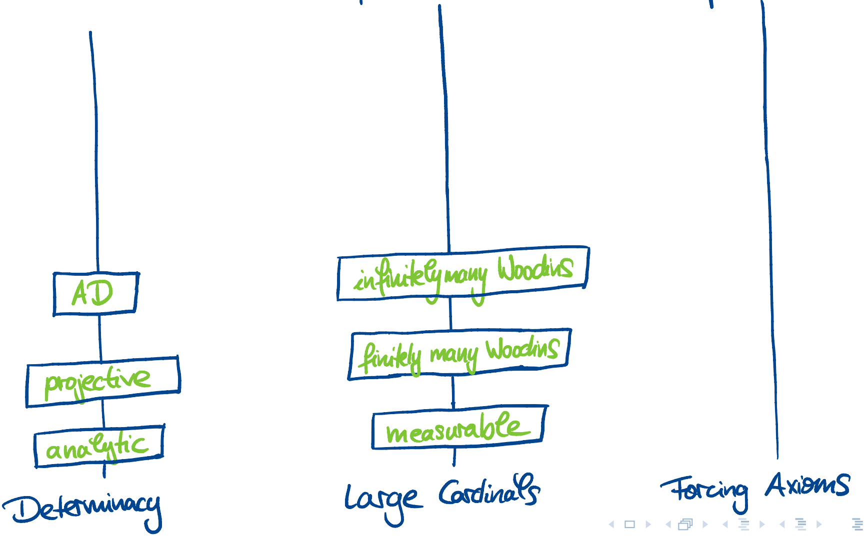

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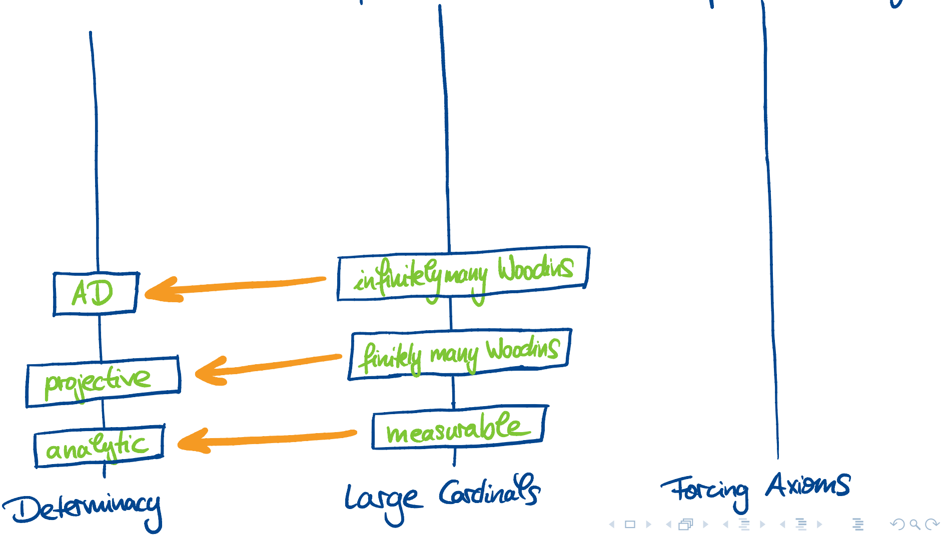

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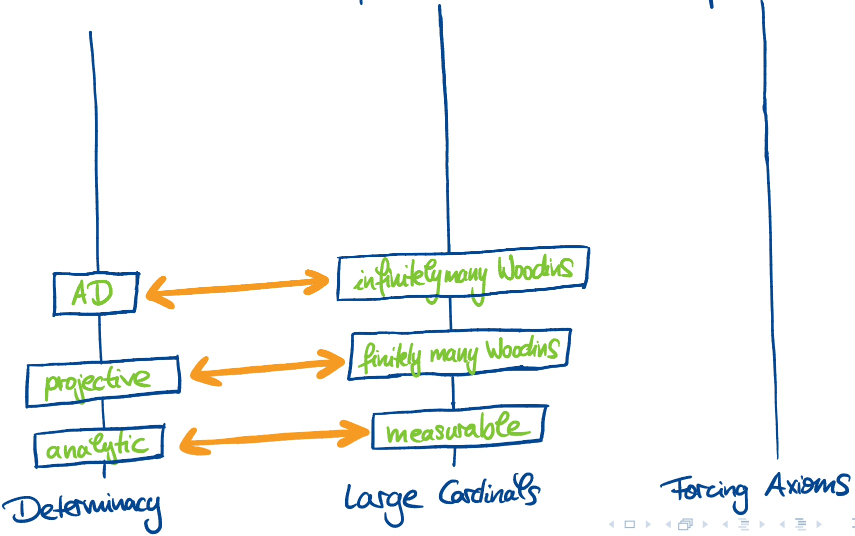

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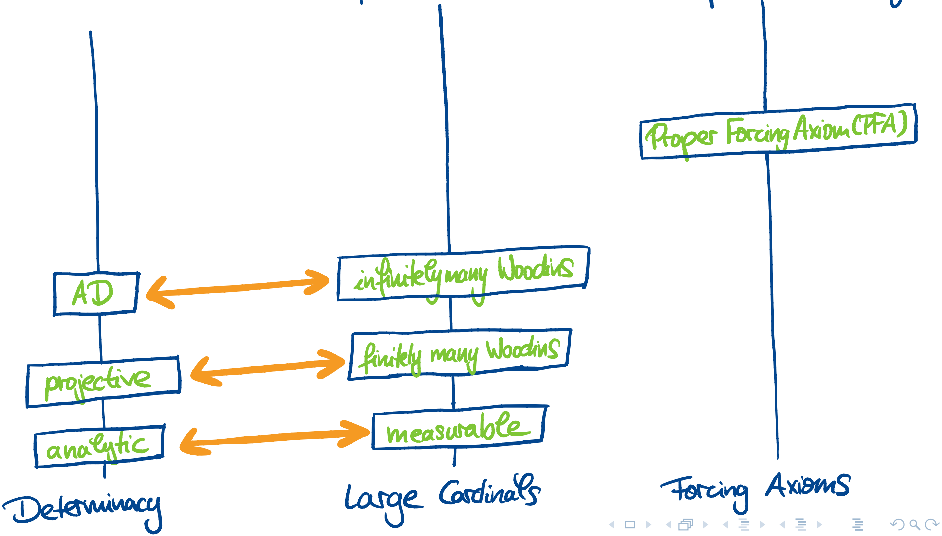

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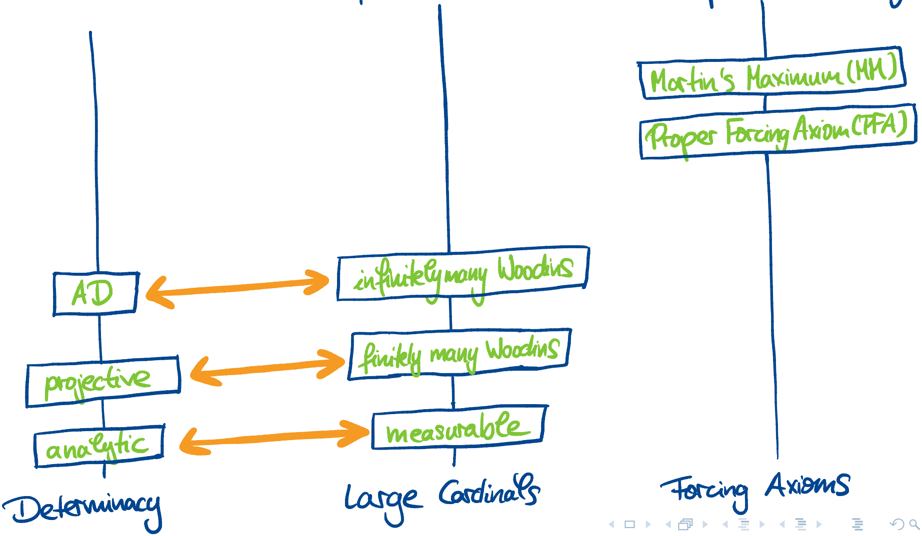

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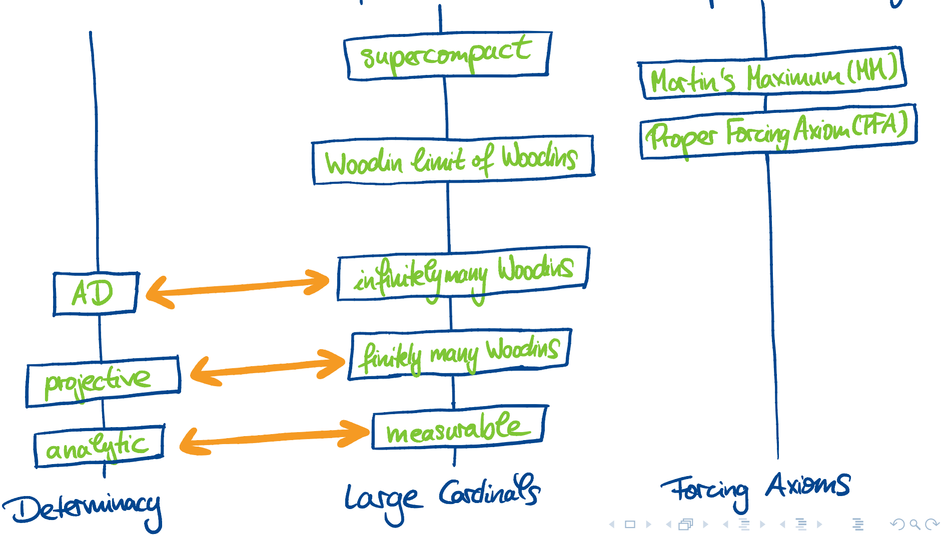

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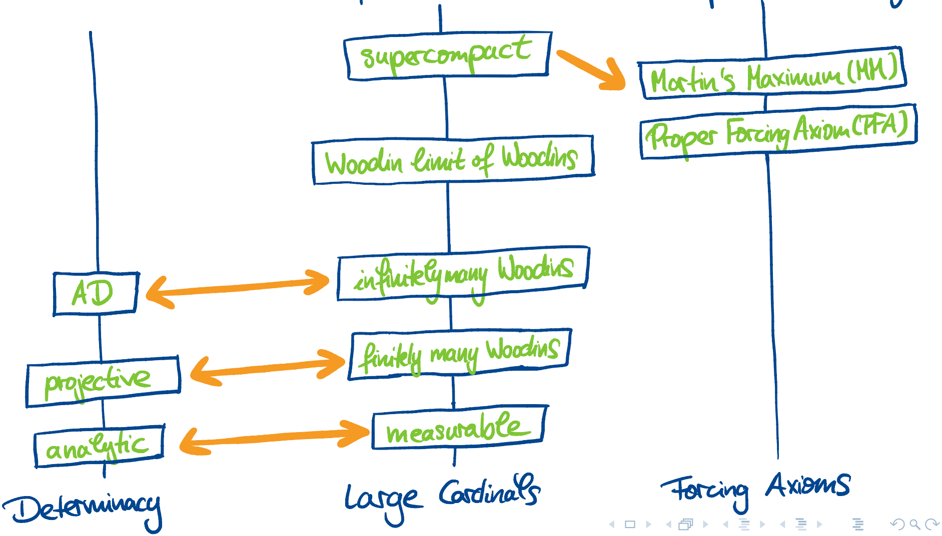

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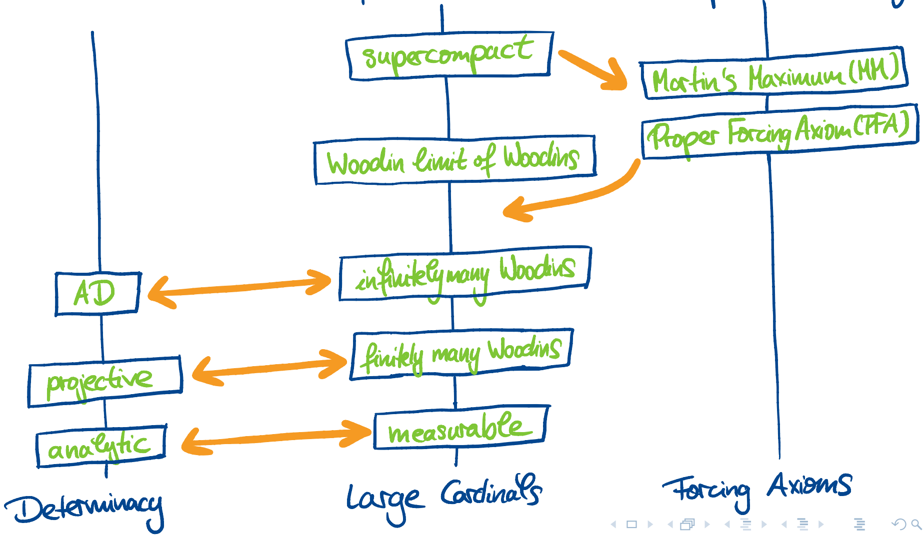

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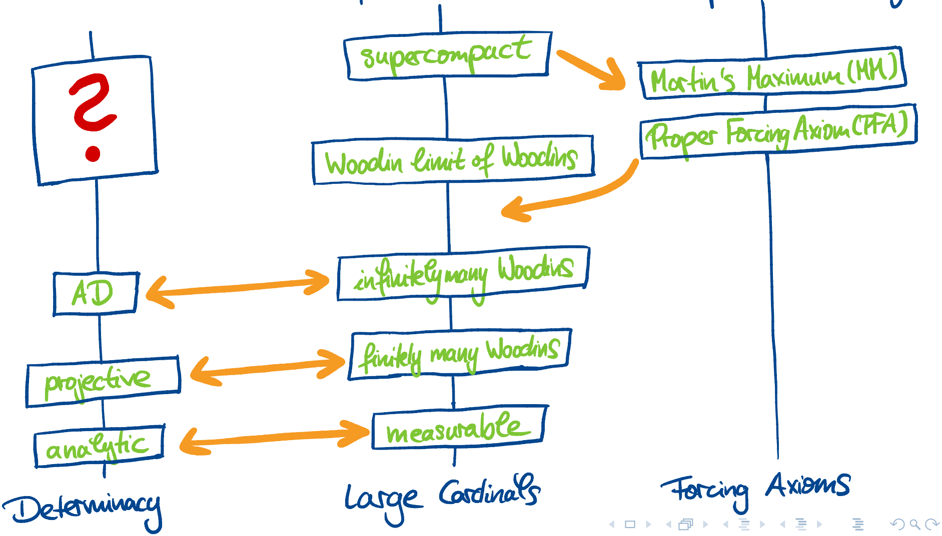

Two scenarios

What axiom(s) could fill the gap in the determinacy hierarchy?

Two scenarios

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Long games

Two scenarios

What axiom(s) could fill the gap in the determinacy hierarchy?

Long games

Strong models